Exercise sheet #10

Exercises marked with a ✎ are to be handed in before **Monday December 2** at noon, in the mailbox at Spiegelgasse 1. Each of these is worth a number of points, as indicated.
Questions marked with a $\star$ are more difficult.

Exercise 1 (Wielandt uniqueness theorem) – Recall that the Gamma function $s \mapsto \Gamma(s)$ is defined on $\mathbb{C}$ by the Weierstrass product

$$\frac{1}{\Gamma(s)} = s\mathrm{e}^{\gamma s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \mathrm{e}^{-s/n}.$$

Let F be a holomorphic function on $\mathrm{Re}(s) > 0$. We assume that

(i) F is bounded in the vertical strip $1 \leqslant \mathrm{Re}(s) \leqslant 2$.

(ii) $\mathrm{F}(s+1) = s\mathrm{F}(s)$ for all s in $\mathrm{Re}(s) > 0$, and $\mathrm{F}(1) = 1$.

The goal of the exercise is to show that F has a meromorphic continuation to $\mathbb{C}$ which coincides with Γ.

1.1. Prove that F has a meromorphic continuation to $\mathbb{C}$. Describe the location of the possible poles of the continuation of F.

1.2. Let $\mathrm{G}(s) := \mathrm{F}(s) - \Gamma(s)$ for all $s \in \mathbb{C}$. Prove that G is holomorphic at $s = 0$, and deduce that G is entire.

1.3. Let $\mathrm{H}(s) = \mathrm{G}(s)\mathrm{G}(1-s)$ for all $s \in \mathbb{C}$. Prove that H is entire and that H is bounded in the vertical strip $0 \leqslant \mathrm{Re}(s) \leqslant 1$.

1.4. Show that $\mathrm{H}(s+1) = -\mathrm{H}(s)$, and compute $\mathrm{H}(0)$.

1.5. Deduce that H is entire and bounded. Conclude that $\mathrm{F} = \Gamma$.

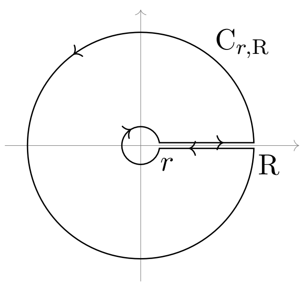
Exercise 2 (Reflection formula for Γ) – Recall that, for all $\mathrm{Re}(s) > 0$, we have

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-u} u^{s-1} \, \mathrm{d}u.$$

Let $s = \sigma$ be such that $0 < \sigma < 1$, and consider the function f_σ defined on $\mathbb{C} \smallsetminus \{0, -1\}$ by $f_\sigma(z) = z^{\sigma-1}/(z+1)$. Here $z \mapsto z^{\sigma-1}$ denotes $z \mapsto \exp((\sigma - 1) \cdot \log z)$, where log is the principal determination of the logarithm.

2.1. Prove that $\Gamma(\sigma)\Gamma(1-\sigma) = \displaystyle\int_0^\infty f_\sigma(x) \, \mathrm{d}x$.

Let $0 < r < 1 < \mathrm{R}$ be real numbers. Denote by $\mathrm{C}_{r,\mathrm{R}}$ the path that starts at $x = r$ on the positive real axis, runs along the real axis to $x = \mathrm{R}$, follows the circle $|z| = \mathrm{R}$ counterclockwise to $x = \mathrm{R}$, runs down the real axis to $x = r$, and follows the circle $|z| = r$ clockwise to $x = r$.



2.2. Prove that f_σ is holomorphic on $\mathrm{C}_{r,\mathrm{R}}$ and inside that contour with the exception of a simple pole at $z = -1$.

2.3. By computing the residue of f_σ at $z = -1$, deduce that $\displaystyle\oint_{\mathrm{C}_{r,\mathrm{R}}} f_\sigma(z)\,\mathrm{d}z = -2\pi i e^{i\pi\sigma}$.

2.4. Prove that $\left|\displaystyle\int_{|z|=\mathrm{R}} f_\sigma(z)\,\mathrm{d}z\right|$ tends to 0 as $\mathrm{R} \to \infty$. Prove that $\left|\displaystyle\int_{|z|=r} f_\sigma(z)\,\mathrm{d}z\right|$ tends to 0 as $r \to 0^+$.

2.5. Deduce from the previous questions that, for all $s = \sigma$ such that $0 < \sigma < 1$, we have

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{1}$$

2.6. Conclude that (1) holds for all $s \in \mathbb{C} \smallsetminus \mathbb{Z}$.

2.7. Prove Euler's formula: for all $s \in \mathbb{C} \smallsetminus \mathbb{Z}$,

$$\sin(\pi s) = \pi s \cdot \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right). \tag{2}$$

Exercise 3 {✎ : 4 points} – For any $x \geqslant 1$, let $\mathrm{F}(x)$ be defined by

$$\mathrm{F}(x) = \sum_{n \leqslant x} \mu(n) \log(x/n),$$

where μ denotes Möbius function.

3.1. Prove that, for any $x \geqslant 1$, we have $\displaystyle\sum_{n \leqslant x} \mathrm{F}(x/n) = \log x$.

3.2. Show that, for any s with $\mathrm{Re}(s) > 1$, we have $\displaystyle\frac{1}{\zeta(s)} = \sum_{n \geqslant 1} \mu(n) n^{-s}$.

3.3. Prove that, for any $s = \sigma + it$ with $\sigma > 1$, we have $|\zeta(s)|^{-1} \leqslant \frac{\zeta(\sigma)}{\zeta(2\sigma)}$.

3.4. For a given $c > 1$, deduce that the integral $\displaystyle\int_{c-i\infty}^{c+i\infty} \zeta(s)^{-1} \cdot x^s \cdot \frac{\mathrm{d}s}{s^2}$ is absolutely convergent.

3.5. Using Perron's formula, prove that for any $c > 1$, and any $x \geqslant 1$ which is not an integer,

$$\mathrm{F}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\zeta(s)} \cdot x^s \cdot \frac{\mathrm{d}s}{s^2}.$$

Exercise 4 (Primes in arithmetic progression) {✎ : 10 points} – Let $q > 1$ be an integer. A group morphism $\chi_q : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ can be extended into a map $\chi : \mathbb{Z} \to \mathbb{C}$ by setting

$$\chi(a) := \begin{cases} 0 & \text{if } \gcd(q,a) > 1, \\ \chi_q(a \bmod q) & \text{otherwise.} \end{cases}$$

The resulting map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo q. We let X_q denote the set of Dirichlet characters modulo q. We say that χ is the trivial Dirichlet character if the underlying character χ_q is the trivial group morphism $x \mapsto 1$.

4.1. Prove that a Dirichlet character modulo q is multiplicative. For $\chi \in \mathrm{X}_q$, show that $|\chi(a)|$ is 0 or 1 for all $a \in \mathbb{Z}$.

4.2. Let $a \in \mathbb{Z}$ be coprime to q, and $n \geqslant 1$. Compute the value of $\sum_{\chi \in \mathrm{X}_q} \overline{\chi(a)} \cdot \chi(n)$.

4.3. Let $\chi \in \mathrm{X}_q$ be a Dirichlet character. Consider the Dirichlet series

$$\mathrm{L}(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Show that the Dirichlet series converges for $\mathrm{Re}(s) > 1$, and give the Euler product for $\mathrm{L}(\chi, s)$.

4.4. If $\chi = \chi_0$ is the trivial Dirichlet character modulo q, relate $\mathrm{L}(\chi_0, s)$ to $\zeta(s)$. Prove that $\mathrm{L}(\chi_0, s)$ extends into a meromorphic function on $\mathrm{Re}(s) > 0$ with a single simple pole at $s = 1$.

4.5. If $\chi \in \mathrm{X}_q$ is non-trivial, show that $\mathrm{L}(\chi, s)$ extends into a holomorphic function on $\mathrm{Re}(s) > 0$. *Hint: use partial summation.*

4.6. Let $\chi \in \mathrm{X}_q$ be non-trivial. Define the series

$$\mathrm{F}(\chi, s) := \sum_{p} \sum_{k=1}^{\infty} \frac{\chi(p^k)}{k} \cdot p^{-ks},$$

where the first sum runs over prime numbers p. Show that the series converges on $\mathrm{Re}(s) > 0$, and that $\exp(\mathrm{F}(\chi, s)) = \mathrm{L}(\chi, s)$.

4.7. Prove that $\mathrm{R}_\chi : s \mapsto \mathrm{F}(\chi, s) - \sum_p \chi(p)p^{-s}$ defines a holomorphic function on $\mathrm{Re}(s) > 1$, which is bounded when $s \to 1^+$.

One can prove that $\mathrm{L}(\chi, 1) \neq 0$ for any non-trivial Dirichlet character $\chi \in \mathrm{X}_q$. In what follows, you may use this fact.

4.8. Consider the product $\mathrm{G}_q(s) := \prod_{\chi \in \mathrm{X}_q} \mathrm{L}(\chi, s)$. For all $s = \sigma > 1$, prove that $\mathrm{G}_q(s)$ is real and $\mathrm{G}_q(s) \geqslant 1$.

4.9. Let s be such that $\mathrm{Re}(s) > 1$. Prove that, for any $a \in \mathbb{Z}$ with $\gcd(q, a) = 1$, we have

$$\frac{1}{\phi(q)} \sum_{\chi \in \mathrm{X}_q} \overline{\chi(a)} \cdot \mathrm{F}(\chi, s) = \sum_{p \equiv a \bmod q} p^{-s} + \mathrm{R}_a(s),$$

where the sum on the right-hand side is over prime numbers which are congruent to a modulo q, and the function $\mathrm{R}_a(s)$ is holomorphic and remains bounded when $s \to 1^+$.

4.10. Deduce Dirichlet's theorem: for any $a \in \mathbb{Z}$ such that $\gcd(a, q) = 1$, there are infinitely many prime numbers p with $p \equiv a \bmod q$.