2.2. Exercises

Exercise 10. — Let $J = (xy, yz, yz)$ in $\overline{k}[x,y,z]$. Find $V = Z(J)$ in $\mathbb{A}^3$. Is it a variety? Is it true that $J = I(Z(J))$? Prove that J cannot be generated by 2 elements.

Let $J' = (xy, (x-y)z) \subset \overline{k}[x,y,z]$. Find $Z(J')$ and compute the radical $\operatorname{rad}(J')$.

Exercise 11. — Let $J = (x^2+y^2-1, y-1) \subset \overline{k}[x,y]$. Find an element $f \in I(Z(J)) \smallsetminus J$.

Exercise 12. — Let $J = (x^2+y^2+z^2, xy+xz+yz) \subset \overline{k}[x,y,z]$. Identify $Z(J)$ and compute $I(V(J))$.

Exercise 13. — Let $f = x^2-y^2$ and $g = x^3+xy^2-y^3-x^2y-x+y$ in $\overline{k}[x,y]$ (assume that the characteristic of k is $\neq 2,3$). Let $W = Z(f,g) \subset \mathbb{A}^2$. Is W an algebraic variety? If not, give a list of affine algebraic varieties V such that $V \subset W$. (*i.e.* give a list of factors of the ideal (f,g)).

Exercise 14. — For any field k, prove that an algebraic set in $\mathbb{A}^1$ is either finite or the whole of $\mathbb{A}^1$. Identify the algebraic varieties among the algebraic sets.

Exercise 15. — Let k be a field.

(a) Let $f, g \in \overline{k}[x,y]$ be irreducible polynomials, not multiples of one another. Prove that $Z(f,g) \subset \mathbb{A}^2$ is finite.

Hint: write $K = \overline{k}(x)$, prove first that f, g have no common factor in the PID $K[y]$. Deduce that there exist $p, q \in K[y]$ such that $pf + qg = 1$. By clearing denominators in p, q, show that there exist $h \in \overline{k}[x]$ and $a, b \in \overline{k}[x,y]$ such that $h = af + bg$. Conclude that there are only finitely many possible values of the x-coordinate of points in $Z(f,g)$.

(b) Prove that an algebraic set $V \subset \mathbb{A}^2$ is a finite union of points and curves. Identify the algebraic varieties among those.

Exercise 16. — In this exercise let $K = \overline{k}$ be the algebraic closure of any field.

(a) Let $f \in K[x_1,\dots,x_n]$ be a nonconstant polynomial (that is $k \notin K$). Prove that $Z(f)$ is a stric subset of $\mathbb{A}^n$.

Hint: suppose that f involves x_n and write $f = \sum_i f_i x_n^i$ where $f_i \in K[x_1,\dots,x_{n-1}]$, use induction on n to conclude.

(b) Let f be as above, suppose that f has degree m in x_n and let $f_m(x_1,\dots,x_{n-1}) \cdot x_n^m$ be its leading term (in x_n). Show that, wherever f_m doesn't vanish, there is a finite nonempty set of points of $Z(f) \subset \mathbb{A}^n$ corresponding to every value of $(x_1,\dots,x_{n-1})$. Deduce that, in particular, $Z(f)$ is infinite for $n \geq 2$.

(c) Putting together the results of the last question and of the previous exercise, show that distinct irreducible polynomials $f, g \in K[x,y]$ define distinct algebraic sets $Z(f)$, $Z(g)$ in $\mathbb{A}^2$.

(d) Can you generalize the results of the last question to $\mathbb{A}^n$?

Exercise 17. — Determine the singular points on the following curves in $\mathbb{A}^2$:

(a) $y^2 = x^3 - x$,

(b) $y^2 = x^3 - 6x^2 + 9x$,

(c) $x^2y^2 + x^2 + y^2 + 2xy(x+y+1) = 0$,

(d) $x^2 = x^4 + y^4$,

(e) $xy = x^6 + y^6$,

(f) $x^3 = y^2 + x^4 + y^4$,

(g) $x^2y + xy^2 = x^4 + y^4$.

Exercise 18. — Show that the hypersurface $X_d \subset \mathbb{P}^n$ defined by $x_0^d + \cdots + x_n^d = 0$ is nonsingular if the characteristic of k does not divide $d \in \mathbb{Z}_{\geq 1}$.

Exercise 19. — Prove that the intersection of a hypersurface $V \subset \mathbb{A}^n$ (that is not a hyperplane) with the tangent hyperplane T_PV to V at $P \in V$ is singular at P.