# Exercise sheet #11

Exercises marked with a ✎ are to be handed in before **Monday December 9** at noon, in the mailbox at Spiegelgasse 1. Each of these is worth a number of points, as indicated.
Questions marked with a $\star$ are more difficult.

**Exercise 1 {✎ : 5 points}** – For all $x \geqslant 1$, we let $\pi(x)$ denote the number of prime numbers $p$ such that $p \leqslant x$. We also let $\Pi(x) := \sum_{p^k \leqslant x} \frac{1}{k}$, where the sum runs over prime powers with $p^k \leqslant x$.

**1.1.** Prove that $\Pi(x) = \sum_{1 \leqslant k \leqslant \log x / \log 2} \frac{\pi(x^{1/k})}{k}$.

**1.2.** Prove that $\Pi(x) = \pi(x) + \mathrm{O}(\sqrt{x} \log x)$ as $x \to \infty$.

**1.3.** Denoting the Riemann zeta function by $\zeta(s)$, prove that $\log \zeta(s)$ is a Dirichlet series that converges absolutely on $\mathrm{Re}(s) > 1$. Here, $\log : \mathbb{C} \to \mathbb{C}$ denotes the principal determination of the complex logarithm (which is real positive on the positive real axis).

**1.4.** For any $c > 1$ and any $x > 1$ which is not an integer, prove that $\Pi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \log \zeta(s) \cdot x^s \frac{\mathrm{d}s}{s}$.

**Exercise 2 (The Dedekind zeta function of $\mathbb{Q}(i)$)** – Let $\mathrm{K} = \mathbb{Q}(i)$ and denote its Dedekind zeta function by $\zeta_{\mathrm{K}}(s)$.

**2.1.** Let $p$ be a prime. Depending on the value of $p \pmod 4$, recall the splitting behaviour of $p$ in K.

**2.2.** For any $s$ with $\mathrm{Re}(s) > 1$, prove that

$$\zeta_{\mathrm{K}}(s) = \zeta(s) \cdot \prod_{\substack{\text{primes } p \\ p \equiv 1 \bmod 4}} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \prod_{\substack{\text{primes } p \\ p \equiv 3 \bmod 4}} \left(1 + \frac{1}{p^s}\right)^{-1}.$$

*Hint: in the Euler product defining $\zeta_{\mathrm{K}}(s)$, group the terms lying above a given rational prime.*

Let $\chi_4 : \mathbb{Z} \to \mathbb{C}$ denote the Dirichlet character modulo 4 defined by $\chi_4(3) = -1$.

**2.3.** Deduce from the above that $\zeta_{\mathrm{K}}(s) = \zeta(s) \cdot \mathrm{L}(\chi_4, s)$.

For any integer $m \geqslant 1$, we let $r(m)$ denote the number of solutions $(u, v) \in \mathbb{Z}^2$ of the equation $m = u^2 + v^2$ *i.e.*, $r(m)$ is the number of representations of $m$ as a sum of two squares.

**2.4.** Prove that, for all $s$ with $\mathrm{Re}(s) > 1$, we have

$$\zeta_{\mathrm{K}}(s) = \frac{1}{4} \sum_{(u,v)} \frac{1}{(u^2 + v^2)^s} = \frac{1}{4} \sum_{m=1}^{\infty} \frac{r(m)}{m^s},$$

where the sum over $(u, v)$ is over pairs $(u, v) \in \mathbb{Z}^2 \smallsetminus \{(0, 0)\}$.

**2.5.** Deduce that $r = 4 \star \chi_4$, where $\star$ denotes the convolution and 4 is the constant function 4 on $\mathbb{Z}$.

**Exercise 3 (Dedekind zeta functions of certain quadratic number fields) {✎ : 5 points}** – Let us fix an odd prime number $q$. We let $q^* := (-1)^{(q-1)/2}q$ and $\mathrm{K} := \mathbb{Q}(\sqrt{q^*})$. Recall that

$$\zeta_{\mathrm{K}}(s) := \sum_{0\neq\mathfrak{a}\subset\mathcal{O}_{\mathrm{K}}} \frac{1}{(\mathrm{N}\,\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(\mathrm{N}\,\mathfrak{p})^s}\right)^{-1}.$$

Also recall the Quadratic Reciprocity Law: For any distinct odd primes $p \neq q$, $\left(\frac{p}{q}\right)\cdot\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$. Moreover, $\left(\frac{2}{q}\right) = (-1)^{(q^2-1)/8}$ for any odd prime $q$, and $\left(\frac{-1}{q}\right) = (-1)^{(q-1)/2}$.

**3.1.** Prove that, for all primes $p$ and suitable $s \in \mathbb{C}$, we have

$$\prod_{\mathfrak{p}|p} \left(1 - \frac{1}{(\mathrm{N}\,\mathfrak{p})^s}\right)^{-1} = \begin{cases} (1-p^{-s})^{-1} & \text{if } p \text{ ramifies in K},\\ (1-p^{-s})^{-2} & \text{if } p \text{ splits in K},\\ (1-p^{-2s})^{-1} & \text{if } p \text{ is inert in K,}\end{cases}$$

the product being over prime ideals $\mathfrak{p}$ of K lying over $p$.

**3.2.** Combine Dedekind-Kummer to the Quadratic Reciprocity Law to show that: for any prime $p$,

$$p \text{ ramifies in K} \Leftrightarrow \left(\frac{p}{q}\right) = 0, \quad p \text{ splits in K} \Leftrightarrow \left(\frac{p}{q}\right) = +1, \quad p \text{ is inert in K} \Leftrightarrow \left(\frac{p}{q}\right) = -1.$$

**3.3.** Let $\chi_q$ denote the Dirichlet character modulo $q$ defined by $a \mapsto \left(\frac{a}{q}\right)$. Deduce from the above that we have $\zeta_{\mathrm{K}}(s) = \zeta(s)\cdot \mathrm{L}(\chi_q, s)$ for all $s \in \mathbb{C}$.

For any $m \geqslant 1$, let $\mathrm{I}_{\mathrm{K}}(m)$ denote the number of integral ideals of K with norm $m$.

**3.4.** Prove that $\mathrm{I}_{\mathrm{K}} = 1 \star \chi_q$ where $\star$ denotes convolution of arithmetic functions, and 1 is the constant function $n \mapsto 1$.

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**Exercise 4 (The Fourier transform of a Gaussian is a Gaussian)** – For any $m \in \mathbb{R}_{>0}$, let $f_m : \mathbb{R} \to \mathbb{R}$ denote the function $x \mapsto \mathrm{e}^{-mx^2}$.

**4.1.** Prove that $f_m$ belongs to the Schwartz class, and that it is the unique solution of the differential equation $y'(x) + 2m\pi x \cdot y(x) = 0$ with $y(0) = 1$.

For any $h \in \mathbb{R}$, recall that $\widehat{f_m}(h) = \int_{\mathbb{R}} f_m(x)\cdot \mathrm{e}^{-2i\pi\cdot hx}\,\mathrm{d}x$.

**4.2.** Prove that $\int_{\mathbb{R}} f_m(x)\,\mathrm{d}x = \widehat{f_m}(0) = \sqrt{\frac{\pi}{m}}$.

**4.3.** Find a differential equation satisfied by $\widehat{f_m}$. Show that $\widehat{f_m} = \sqrt{\frac{\pi}{m}}\cdot f_{m'}$ where $m' = \pi^2/m$.

Let $\mathrm{M} \in \mathcal{M}_{n,n}(\mathbb{R})$ be a real symmetric positive definite $n\times n$ matrix. Define a function $f_{\mathrm{M}} : \mathbb{R}^n \to \mathbb{R}$ by $\mathbf{x} = (x_1, \ldots, x_n) \mapsto \exp(-\mathbf{x}^{\mathrm{T}}\cdot \mathrm{M}\cdot\mathbf{x})$.

**4.3.** Prove that the Fourier transform $\mathbf{h} \mapsto \widehat{f_{\mathrm{M}}}(\mathbf{h}) = \int_{\mathbb{R}^n} f_{\mathrm{M}}(\mathbf{y})\cdot \mathrm{e}^{-2i\pi\cdot\mathbf{h}^{\mathrm{T}}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{y}$ is well-defined on $\mathbb{R}^n$.

**4.4.** Show that there exist a diagonal matrix $\Lambda$ with positive diagonal entries $\lambda_1, \ldots, \lambda_n$, and an orthogonal matrix P such that $\mathrm{P}^{\mathrm{T}}\cdot\mathrm{M}\cdot\mathrm{P} = \Lambda$.

**4.5.** For any $\mathbf{h} \in \mathbb{R}^n$, we let $\mathbf{g} := \mathrm{P}^{\mathrm{T}}\cdot\mathbf{h} = (g_1, \ldots, g_n)$. Prove that

$$\int_{\mathbb{R}^n} f_{\mathrm{M}}(\mathbf{y})\cdot \mathrm{e}^{-2i\pi\cdot\mathbf{h}^{\mathrm{T}}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{y} = \prod_{j=1}^{n}\left(\int_{\mathbb{R}} \mathrm{e}^{-\lambda_j z_j^2}\cdot \mathrm{e}^{-2i\pi\cdot g_j z_j}\,\mathrm{d}z_j\right).$$

*Hint: change variables via* $\mathbf{y} = \mathrm{P}\cdot\mathbf{z}$.

**4.6.** Conclude that $\widehat{f_{\mathrm{M}}} = c(\mathrm{M})\cdot f_{\mathrm{M}'}$, where $\mathrm{M}' \in \mathcal{M}_{n,n}(\mathbb{R})$ and $c(\mathrm{M})$ are to be computed in terms of M.

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