Exercise sheet #12

Exercises marked with a ✎ are to be handed in before **Tuesday December 17** at noon, in the mailbox at Spiegelgasse 1. Each of these is worth a number of points, as indicated.
Questions marked with a $\star$ are more difficult.

Exercise 1 (Dirichlet density and Bauer's theorem) – Let S be a set of prime numbers. We say that S has a Dirichlet density if the limit

$$\lim_{\sigma\to 1^+}\left(\sum_{p\in \mathrm{S}} p^{-\sigma}\right)\cdot\left(\sum_{p} p^{-\sigma}\right)^{-1}.$$

exists. The sum in the denominator is over the set of all prime numbers. In case the limit exists, we denote it by $\delta(\mathrm{S})$ and call it the Dirichlet density of S.

Let K be a number field. For any prime number p, we let $\omega_{\mathrm{K}}(p)$ be the number of prime ideals $\mathfrak{p}$ of K which lie above p and have residual degree $f_{\mathfrak{p}} = 1$.

1.1. For any $\sigma \in (1, +\infty)$, show that $\log \zeta_{\mathrm{K}}(\sigma) = \displaystyle\sum_{k=1}^{\infty}\sum_{p}\frac{p^{-k\sigma}}{k}\cdot\left(\sum_{\substack{\mathfrak{p}\mid p\\ f_{\mathfrak{p}}\mid k}} f_{\mathfrak{p}}\right)$.

1.2. Deduce that there exists a constant $c_0 > 0$ (which does not depend on K) such that, for all $\sigma > 1$, we have $\displaystyle\left|\log\zeta_{\mathrm{K}}(\sigma) - \sum_{p}\frac{\omega_{\mathrm{K}}(p)}{p^{\sigma}}\right| \leqslant c_0\cdot[\mathrm{K}:\mathbb{Q}]$.

1.3. Deduce that, as $\sigma > 1$ tends to 1^+, we have $\displaystyle\frac{\log\zeta_{\mathrm{K}}(\sigma)}{\log\zeta(\sigma)} \sim \frac{\sum_p p^{-\sigma}\cdot\omega_{\mathrm{K}}(p)}{\sum_p p^{-\sigma}} \xrightarrow[\sigma\to 1^+]{} 1.$

Let $y \in \mathbb{Z}_{\geqslant 1}$ be given. Consider the set $\mathrm{S}_{y,\mathrm{K}}$ consisting of prime numbers p which have at least y distinct prime (ideal) divisors $\mathfrak{p}$ in K with $f_{\mathfrak{p}} = 1$.

1.4. Assuming that $\mathrm{S}_{y,\mathrm{K}}$ admits a Dirichlet density $\delta(\mathrm{S}_{y,\mathrm{K}})$, prove that $\delta(\mathrm{S}_{y,\mathrm{K}}) \leqslant 1/y$.

1.5. Prove that K has infinitely many prime ideals with residual degree $f_{\mathfrak{p}} = 1$.

Assume now that the number field K/$\mathbb{Q}$ is Galois. Let $\mathcal{S}_{\mathrm{K}}$ be the set of prime numbers p which split completely in K.

1.6. Let p be a prime. Prove that p splits completely if and only if $\omega_{\mathrm{K}}(p) > 0$, if and only if $\omega_{\mathrm{K}}(p) = n$.

1.7. Show that $\mathcal{S}_{\mathrm{K}}$ has a Dirichlet density and that $\delta(\mathcal{S}_{\mathrm{K}}) = 1/[\mathrm{K}:\mathbb{Q}]$. (In particular, $\mathcal{S}_{\mathrm{K}}$ is infinite.)

In the last questions, we prove Bauer's theorem. Let K and L be two Galois number fields. As in the previous questions, we write $\mathcal{S}_{\mathrm{K}}$ and $\mathcal{S}_{\mathrm{L}}$ for the set of prime numbers which are completely split in K and L, respectively. We assume that $\mathcal{S}_{\mathrm{K}} = \mathcal{S}_{\mathrm{L}}$, and we will deduce that K = L.

1.8. Using the previous question, prove that $[\mathrm{K}:\mathbb{Q}] = [\mathrm{L}:\mathbb{Q}]$.

1.9. Consider the compositum $\mathrm{M} := \mathrm{K}\cdot\mathrm{L}$ (*i.e.* the smallest extension of $\mathbb{Q}$ containing both K and L). Show that $\delta(\mathcal{S}_{\mathrm{K}}) \leqslant [\mathrm{M}:\mathbb{Q}]^{-1}$. *Hint: show that the primes in* $\mathcal{S}_{\mathrm{K}}$ *split completely in* M, *use* ***2.4***.

1.10. Conclude that M = K = L.

Exercise 2 (Closed formula for $\mathrm{L}(\chi, 1)$) {✐ : 8 points} – Let $q \geqslant 2$ be an integer, and let X_q denote the set of Dirichlet characters modulo q. Recall that, for each non-trivial $\chi \in \mathrm{X}_q$, we have defined a Dirichlet series $\mathrm{L}(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ which converges on $\mathrm{Re}(s) > 0$.

The goal of the exercise is to give a closed formula for $\mathrm{L}(\chi, 1)$.

2.1. For any $\theta \in (0, 2\pi)$, we put $\mathrm{L}(\theta) = \displaystyle\sum_{n=1}^{\infty} \frac{\exp(in\theta)}{n}$. Prove that the series converges.

2.2. Prove that $\mathrm{L}(\theta) = -\log\left(2\sin\frac{\theta}{2}\right) + i\frac{\pi-\theta}{2}$. Here, log denotes the principal branch of the complex logarithm.

We now fix a non-trivial Dirichlet character $\chi \in \mathrm{X}_q$. Let $\mathrm{G}(\chi) := \displaystyle\sum_{x=1}^{q-1} \overline{\chi}(x) \cdot \exp\left(\frac{2\pi i \cdot x}{q}\right)$.

2.3. Show that, for all $n \in \mathbb{Z}$ which are coprime to q, we have:

$$\chi(n) = \frac{1}{\mathrm{G}(\chi)} \sum_{y \in (\mathbb{Z}/q\mathbb{Z})^{\times}} \overline{\chi}(y) \cdot \exp\left(\frac{2\pi i \cdot ny}{q}\right). \tag{1}$$

Hint : note that $\chi(y^{-1}) = \overline{\chi}(y)$ for all $y \in (\mathbb{Z}/q\mathbb{Z})^{\times}$.

For the rest of the exercise, we assume that $\chi \in \mathrm{X}_q$ is *primitive* modulo q (*i.e.* χ is not induced by a Dirichlet character $\chi' \in \mathrm{X}_{q'}$ modulo some strict divisor q' of q). Among other things, this implies that equality (1) holds for all $n \in \mathbb{Z}$. *You don't have to prove this.*

2.4. For a primitive character $\chi \in \mathrm{X}_q$, prove by using (1) that

$$\mathrm{L}(\chi, 1) = \frac{-1}{\mathrm{G}(\chi)} \cdot \left(\sum_{y=1}^{q-1} \overline{\chi}(y) \cdot \log\sin\frac{\pi y}{q} + \frac{\pi i}{q} \cdot \sum_{y=1}^{q-1} \overline{\chi}(y) \cdot y \right).$$

2.5. Assume moreover that $\chi(-1) = 1$ (one says that χ is *even*). Prove that $\sum_{y=1}^{q-1} \overline{\chi}(y) \cdot y = 0$ in this case, and deduce that

$$\mathrm{L}(\chi, 1) = \frac{-1}{\mathrm{G}(\chi)} \cdot \sum_{y=1}^{\mathrm{N}-1} \overline{\chi}(y) \cdot \log\sin\frac{\pi y}{q}.$$

2.6. Assume now that $\chi(-1) = -1$ (one says that χ is *odd*). Prove that $\sum_{y=1}^{q-1} \overline{\chi}(y) \cdot \log\sin\frac{\pi y}{q} = 0$ under this assumption, and deduce that

$$\mathrm{L}(\chi, 1) = \frac{-\pi i}{q\mathrm{G}(\chi)} \cdot \sum_{y=1}^{q-1} \overline{\chi}(y) \cdot y.$$

2.7. As a first application, consider the following situation. Let $q = 4$ and $\chi_4 \in \mathrm{X}_4$ denote the Dirichlet character defined by $\chi_4(-1) = -1$. Deduce from the above relations that $\mathrm{L}(\chi_4, 1) = \dfrac{\pi}{4}$.

2.8. As a second application, consider the case where $q = 5$ and $\chi_5 \in \mathrm{X}_5$ is the Dirichlet character modulo 5 defined by $\chi_5(-1) = \chi_5(1) = 1$, and $\chi_5(2) = \chi_5(3) = -1$. Deduce from the above that

$$\mathrm{L}(\chi_5, 1) = \frac{\log\eta}{\sqrt{5}}, \quad \text{where} \quad \eta = \frac{\sin\frac{2\pi}{5} \cdot \sin\frac{3\pi}{5}}{\sin\frac{\pi}{5} \cdot \sin\frac{4\pi}{5}}.$$

2.9. Consider the number field $\mathrm{K} = \mathbb{Q}(\sqrt{5})$. Compute its discriminant, its class number, its fundamental unit and the number of roots of unity in K. Comment on the equality proven in **1.8**.

Exercise 3 (The different) {✎ : 4 points} – Let K be a number field with discriminant d_{K}. We denote by $\mathrm{Tr}_{\mathrm{K}} : \mathrm{K} \to \mathbb{Q}$ the trace of $\mathrm{K}/\mathbb{Q}$. Recall that the Dedekind dual of a fractional ideal $\mathfrak{b}$ of K is defined by

$$\mathfrak{b}' = \{\alpha \in \mathrm{K} : \forall \beta \in \mathfrak{b},\ \mathrm{Tr}_{\mathrm{K}}(\alpha\beta) \in \mathbb{Z}\}.$$

3.1. Prove that the Dedekind dual $\mathfrak{b}'$ of a fractional ideal $\mathfrak{b}$ of K is also a fractional ideal of K. Moreover, check that $(\mathfrak{b}')' = \mathfrak{b}$.

We define the different $\mathfrak{d}_{\mathrm{K}}$ of K by the equality $\mathfrak{d}_{\mathrm{K}}^{-1} := (\mathcal{O}_{\mathrm{K}})'$.

3.2. Prove that $\mathfrak{d}_{\mathrm{K}}$ is an (integral) ideal of K.

3.3. Check that, for any fractional ideal $\mathfrak{b}$ of K, we have $\mathfrak{b}' = \mathfrak{b}^{-1} \cdot \mathfrak{d}_{\mathrm{K}}^{-1}$.

3.4. Prove the equality $\mathrm{N}(\mathfrak{d}_{\mathrm{K}}) = |d_{\mathrm{K}}|$.