A parallelogram height inequality for Drinfeld modules

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Abstract – We prove inequalities relating the Taguchi heights, respectively the graded heights, of four Drinfeld modules arranged in a "parallelogram of isogenies". This inequality is the analogue for Drinfeld modules of the parallelogram inequality of Rémond [Rém22] for abelian varieties over number fields and of Griffon–Le Fourn–Pazuki [GLP25] for abelian varieties over function fields.

Keywords: Drinfeld modules, Isogenies, Graded height, Taguchi height.

2020 Math. Subj. Classification: Primary 11G09, 11G50, 14G40, 14K02; Secondary 11J93.

Introduction

Fix a finite field $\mathbb{F}_q$, and let $K_0 = \mathbb{F}_q(T)$ denote the rational function field over $\mathbb{F}_q$. In this article, we consider Drinfeld $\mathbb{F}_q[T]$-modules ϕ defined over a finite field extension K of K_0. There are several notions of heights attached to these objects: we focus on the graded height $h_{\mathrm{Gr},K}(\cdot)$ and the Taguchi height $h_{\mathrm{Tag},K}(\cdot)$. We give precise definitions of these two heights in section 4 below. This choice is motivated by the growing relevance of these two heights in Diophantine problems related to Drinfeld modules. See, for instance, [Wei20] where Wei gives a formula "à la Colmez" for the stable Taguchi height of a CM Drinfeld module over K. Moreover, in a recent paper [BPR21] Breuer, Pazuki, and Razafinjatovo gave bounds on the variation of the graded height by isogeny in this context.

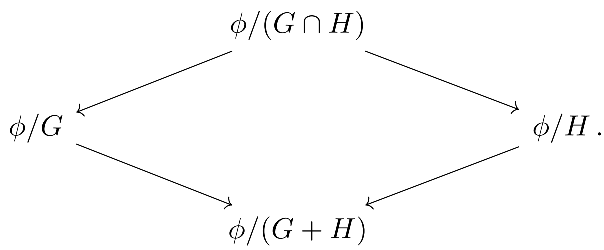
Let ϕ be a Drinfeld module of rank $r \geq 1$ defined over K. To any isogeny $f : \phi \to \phi'$ of Drinfeld modules, one associates a finite dimensional $\mathbb{F}_q$-vector space $\ker(f) \subset \overline{K}$ called the kernel of f. We can then think of the target Drinfeld module ϕ' as "the quotient" $\phi/\ker(f)$. We let $\mathtt{SG}_K(\phi)$ denote the set of all kernels of isogenies $f : \phi \to \phi'$ between ϕ and other Drinfeld modules ϕ' defined over K. We refer to section 1.2 for precise definitions. With this notation at hand, we can state our main result:

Theorem A. *Let K be a finite extension of $K_0 = \mathbb{F}_q(T)$. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank $r \geq 1$ defined over K, and let $G, H \in \mathtt{SG}_K(\phi)$.*

(1) We have
$$h_{\mathrm{Gr},K}(\phi/(G \cap H)) + h_{\mathrm{Gr},K}(\phi/(G + H)) \leq h_{\mathrm{Gr},K}(\phi/G) + h_{\mathrm{Gr},K}(\phi/H).$$

(2) If ϕ has stable reduction at all finite places of K, then
$$h_{\mathrm{Tag},K}(\phi/(G \cap H)) + h_{\mathrm{Tag},K}(\phi/(G + H)) \leq h_{\mathrm{Tag},K}(\phi/G) + h_{\mathrm{Tag},K}(\phi/H).$$

These two relations are called "parallelogram inequalities" because of the shape of the diagram of isogenous Drinfeld modules that one constructs from the data ϕ, G, H of the theorem:



Arrows in this picture represent isogenies of Drinfeld modules. We underline the fact that Theorem A involves no extraneous terms (e.g. a dependence on the size of G, H), contrary to the bounds given in [BPR21] on the variation these heights in isogeny classes.

It is also interesting to note the perfect similarity between Theorem A and a very recent result of [GLP25] in the context of abelian varieties defined over K:

Theorem (Theorem 7.1 in [GLP25])**.** *Let B be a semi-stable abelian variety defined over K. For any finite subgroup schemes G, H of B, we have*

$$\mathrm{h}_{\mathrm{diff}}(B/(G\cap H)) + \mathrm{h}_{\mathrm{diff}}(B/(G+H)) \leq \mathrm{h}_{\mathrm{diff}}(B/G) + \mathrm{h}_{\mathrm{diff}}(B/H),$$

where $\mathrm{h}_{\mathrm{diff}}(\cdot)$ denotes the differential height over K.

This result was itself inspired by the parallelogram inequality of Rémond (Théorème 1.1 in [Rém22]) in the setting of abelian varieties over a number field, where the rôle of the differential height is played by the Faltings height. The proofs of the respective parallelogram inequalities in these three contexts are based on different strategies. With Theorem A, the classical Diophantine triptych "abelian varieties over number fields" – "abelian varieties over function fields" – "Drinfeld modules" is thus complete as regards the parallelogram inequality.

Both the graded and the Taguchi heights are defined as sums over places v of K of local contributions $h^v_{\star,K}(\cdot)$. Theorem A is actually a consequence of parallelogram inequalities for these local contributions. Specifically, we prove the following result.

Theorem B. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank $r \geq 1$ defined over K, and let $G, H \in \mathtt{SG}_K(\phi)$. Let v be a place of K, and let $\star \in \{\mathrm{Tag}, \mathrm{Gr}\}$. If v is finite and $\star = \mathrm{Tag}$, we assume that ϕ has stable reduction at v. Then*

$$h^v_{\star,K}(\phi/(G\cap H)) + h^v_{\star,K}(\phi/(G+H)) \leq h^v_{\star,K}(\phi/G) + h^v_{\star,K}(\phi/H). \tag{$\diamond$}$$

In case $\star = \mathrm{Gr}$, we prove ($\diamond$) for an arbitrary place v in Theorem 5.3. The latter is mainly based on a relation (Theorem 3.3), proved in section 3, between valuation polygons of polynomials with coefficients in a valued field. For $\star = \mathrm{Tag}$, inequality ($\diamond$) is proved in Corollary 5.4 for finite v, and in Theorem 5.2 if v is infinite. For $\star = \mathrm{Tag}$ and v infinite, it is worth noting that inequality ($\diamond$) is actually an equality. The key technical result (Theorem 2.6) in this case, which we prove in section 2, is a relation between covolumes of A-lattices. In the process of obtaining our main results, we also reprove a general statement of independent interest, to the effect that isogenies preserve stable reduction at a place (Proposition 5.1).

We close this introduction by spelling out a concrete special case of Theorem A for Drinfeld modules of rank 2. One classically associates to a Drinfeld module $\varphi : A \to K_0\{\tau\}$ of rank 2 defined over $K_0 = \mathbb{F}_q(T)$ its j-invariant $j(\varphi) \in \mathbb{F}_q(T)$. The *modular height* of φ is then the degree of $j(\varphi)$ as a rational function in T. In the context of elliptic curves, Vélu's formulas (see [Vél71]) give explicit relations between the j-invariants of isogenous elliptic curves in terms of points in the kernel of the isogeny. In contrast, there seems to be no analogous formulas in the Drinfeld setting. Nonetheless, our Theorem A imposes a weak relation between the j-invariants of Drinfeld modules of rank 2 in an isogeny parallelogram (see section 5.6):

Corollary C. *Let ϕ be a Drinfeld module of rank 2 defined over $\mathbb{F}_q(T)$ and $G, H \in \mathtt{SG}_K(\phi)$. Then*

$$\deg j(\phi/(G\cap H)) + \deg j(\phi/(G+H)) \leq \deg j(\phi/G) + \deg j(\phi/H).$$

General notation – We fix a finite field $\mathbb{F}_q$ with cardinality q. We write $A \coloneqq \mathbb{F}_q[T]$ for the ring of polynomials with coefficients in $\mathbb{F}_q$, and $K_0 \coloneqq \mathbb{F}_q(T)$ for the fraction field of A. We let K denote a finite field extension of K_0 *i.e.*, K is a function field containing $\mathbb{F}_q$.

For any such field K, we let M_K denote the set of places of K. Each place $v \in M_K$ corresponds to a non-archimedean valuation $\mathrm{ord}_v : K \to \mathbb{Z} \cup \{\infty\}$.

The field K_0 has a distinguished place, denoted by ∞, which is characterized by the fact that $\mathrm{ord}_\infty(a) = \deg(a)$ for any $a \in A$. The other places of K_0 are called finite places: each finite place of K_0 corresponds to a monic irreducible polynomial $P \in A$ and, for any $a \in A$, $\mathrm{ord}_v(a)$ is the multiplicity of P as a divisor of a.

A place $v \in M_K$ is called infinite if v extends the place $\infty \in M_{K_0}$, and v is called finite otherwise. We let M_K^∞ (resp. M_K^f) denote the set of infinite (resp. finite) places of K. If a place $v \in M_K$ extends a place $w \in M_{K_0}$, we normalize ord_v so that $\mathrm{ord}_v(K^*) = \mathbb{Z}$. For any place $v \in M_K$, we write K_v for the completion of K at v. To any place $v \in M_K$ lying above a place $w \in M_{K_0}$, we then associate its ramification index e_v, its residual degree f_v, and its local degree $n_v = [K_v : K_{0,w}]$. Recall that $n_v = e_v\, f_v$, see for instance Proposition 1.2.11 in [BG06]. Our choices of normalization are consistent with those in [Pap23, Gos12, DD99], but differ from those in [BPR21].

1. Isogenies between Drinfeld modules

1.1. Arithmetic of twisted polynomial rings. – Let F be a field containing $\mathbb{F}_q$, and let F^{sep} denote the separable closure of F in a chosen algebraic closure $\overline{F}$ of F. We gather various useful facts about the ring $F\{\tau\}$ of twisted polynomials in τ: addition in $F\{\tau\}$ is the usual addition of polynomials and multiplication is given by

$$\left(\sum_i f_i\,\tau^i\right)\cdot\left(\sum_j g_j\,\tau^j\right) = \sum_k \left(\sum_{i+j=k} f_i\, g_j^{q^i}\right)\tau^k.$$

The map that sends τ to $\mathrm{Frob}_q : x \mapsto x^q$ induces a $\mathbb{F}_q$-algebra isomorphism between $F\{\tau\}$ and the $\mathbb{F}_q$-algebra (endowed with composition) of endomorphisms of the additive group scheme $\mathbb{G}_{a,F}$ over F. A polynomial $f \in F\{\tau\}$ will be called *normalized* if its constant coefficient is 1.

For any $f = f_0\,\tau^0 + f_1\,\tau + \cdots + f_d\,\tau^d \in K\{\tau\}$, we let

$$f(x) \coloneqq f_0\, x + f_1\, x^q + \cdots + f_d\, x^{q^d} \in F[x].$$

The map $x \mapsto f(x)$ is the $\mathbb{F}_q$-linear endomorphism of $\mathbb{G}_{a,F}$ associated to f. The derivative (with respect to x) of $f(x)$ is the constant polynomial f_0. When $f_0 \neq 0$, the polynomial $f(x)$ is thus separable, *i.e.*, has distinct roots in $\overline{F}$ (see [Pap23, p. 137]). It is, in particular, the case if f is normalized.

Definition 1.1. For any non-zero separable polynomial $f \in F\{\tau\}$, we define the *kernel of* f to be the set of roots in $\overline{F}$ of the corresponding polynomial $f(x) \in F[x]$:

$$\ker(f) \coloneqq \left\{\beta \in \overline{F} \mid f(\beta) = 0\right\} \subset \overline{F}.$$

Since $x \mapsto f(x)$ is a $\mathbb{F}_q$-linear map, $\ker(f)$ is a finite dimensional $\mathbb{F}_q$-vector space. Note that $|\ker(f)| = \deg_x f(x) = q^{\deg_\tau f}$ and therefore $\dim_{\mathbb{F}_q} \ker(f) = \deg_\tau f$. Conversely:

Lemma 1.2. *Let G be a finite dimensional $\mathbb{F}_q$-vector space contained in F^{sep}, which is stable under the action of* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. *Then there is a unique normalized polynomial $f \in F\{\tau\}$ such that* $\ker(f) = G$.

Proof. Given G as in the statement and a non-zero constant $c \in F^*$, we let

$$f_{G,c}(x) \coloneqq c \prod_{\alpha\in G}(x-\alpha) \in F^{\mathrm{sep}}[x].$$

Since $G \subset F^{\mathrm{sep}}$ is (globally) stable under the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$, the polynomial $f_{G,c}(x)$ has coefficients in F. The fact that G is a $\mathbb{F}_q$-vector space allows to show that $f_{G,c}(x) \in F[x]$ is $\mathbb{F}_q$-linear *i.e.*, that $f_{G,c}(x+\alpha\, y) = f_{G,c}(x) + \alpha\, f_{G,c}(y)$ for any $x, y \in \overline{F}$ and $\alpha \in \mathbb{F}_q$ (see the proof of [Pap23, Proposition 3.3.11]). Hence $f_{G,c}(x)$ comes from a non-zero element $f_{G,c} \in F\{\tau\}$.

It is clear that any separable polynomial $f \in F\{\tau\}$ with kernel G is of the form $f_{G,c}$ for some $c \in F^*$. There is exactly one choice of $c \in F^*$ such that $f_{G,c}$ has constant coefficient 1, namely $c = \prod_{\beta\in G\smallsetminus\{0\}}(-\beta)^{-1}$. □

Let $d, f \in F\{\tau\}$. One says that d divides f (on the right) if there exists $g \in F\{\tau\}$ such that $f = g \cdot d$. Equivalently, d divides f if and only if f belongs to the left-ideal $F\{\tau\} \cdot d$.

A classical argument shows that the ring $F\{\tau\}$ admits a right-division algorithm (see [Pap23, Theorem 3.1.3] or [Gos12, Proposition 1.6.2], for instance), which immediately implies that any left-ideal in $F\{\tau\}$ is principal.

Lemma 1.3. *Let $d, f \in F\{\tau\} \setminus \{0\}$ be separable polynomials. Then d divides f if and only if $\ker(d) \subset \ker(f)$.*

Proof. It is clear that $\ker(d) \subset \ker(f)$ if d divides f. Assume conversely that $\ker(d) \subset \ker(f)$. By the right-division algorithm in $F\{\tau\}$, there exists a unique pair $g, r \in F\{\tau\}$ such that $f = g \cdot d + r$ and $\deg r < \deg d$ or $r = 0$. Since $r(\beta) = f(\beta) - g(d(\beta)) = 0$ for all $\beta \in \ker(d)$, the polynomial $r(x) \in F[x]$ has at least $|\ker(d)|$ roots in $\overline{F}$. Since d is separable, $|\ker(d)| = q^{\deg_\tau d}$. Having this many roots forces $r(x)$ to be 0. □

Given two non-zero polynomials $f, g \in F\{\tau\}$, it makes sense to define their (right)-GCD as the unique normalized generator of the ideal $F\{\tau\} \cdot f + F\{\tau\} \cdot g$, and their (right)-LCM as the unique normalized generator of the ideal $F\{\tau\} \cdot f \cap F\{\tau\} \cdot g$. Note that our terminology is non-standard, [Gos12, §1.6] requires that the GCD be *monic* instead.

Lemma 1.4. *Let $f, g \in F\{\tau\}$ be non-zero separable polynomials. Write $d \in F\{\tau\}$ for their GCD and $\ell \in F\{\tau\}$ for their LCM. Then*

$$\ker(d) = \ker(f) \cap \ker(g) \qquad \textit{and} \qquad \ker(\ell) = \ker(f) + \ker(g).$$

Proof. This follows from the definitions of GCD and LCM as well as the previous lemma. One can indeed rephrase the definition of the LCM as "the LCM of f and g is the unique normalized polynomial ℓ of smallest degree such that $\ker(\ell)$ contains both $\ker(f)$ and $\ker(g)$". A similar reformulation can be done for the GCD. □

1.2. Drinfeld modules and isogenies. – Let F be a field extension of K_0. The field F is then equipped with a (trivial) A-module structure *via* the inclusion $A \hookrightarrow K_0 \hookrightarrow F$. In the terminology of [Pap23, §3.2], F is an A-field of A-characteristic 0.

We recall the definition of a Drinfeld module in this context:

Definition 1.5. A Drinfeld module of rank $r \geq 1$ defined over F is a $\mathbb{F}_q$-algebra homomorphism $\phi : A \to F\{\tau\}$, $a \mapsto \phi_a$ such that for each $a \in A$,

- the constant coefficient of $\phi_a \in F\{\tau\}$ is $a \in A \subset F$,
- and $\deg_\tau \phi_a = r \deg_T a$.

A Drinfeld module ϕ is entirely determined by the data $\phi_T \in F\{\tau\}$ because the $\mathbb{F}_q$-algebra $A = \mathbb{F}_q[T]$ is generated by T. We refer to [Ros02, Chap. 12], [Gos12, Chap. 4] or [Pap23, Chap. 3] for more details about the basics of the arithmetic theory of Drinfeld modules.

Definition 1.6. Given two Drinfeld modules ϕ, ψ defined over F, a morphism from ϕ to ψ is a polynomial $f \in F\{\tau\}$ such that

$$f \cdot \phi_a = \psi_a \cdot f \text{ for all } a \in A. \tag{1.1}$$

Such a morphism will be denoted by $f : \phi \to \psi$. A morphism is called an *isogeny* if it is non-zero. We say that an isogeny is *normalized* if its constant coefficient is 1.

If there exists an isogeny between two Drinfeld modules, they will be called *isogenous*.

Since A is generated by T as a $\mathbb{F}_q$-algebra and since ϕ, ψ are $\mathbb{F}_q$-algebra morphisms, the "commutation" condition (1.1) is equivalent to simply requiring that $f \cdot \phi_T = \psi_T \cdot f$.

If there exists an isogeny $f : \phi \to \psi$, then ϕ and ψ have the same rank (see [Pap23, Proposition 3.3.4(1)]). Moreover, in this case there exists a (non unique) *dual isogeny* $\widehat{f} : \psi \to \phi$ (see [Pap23, Proposition 3.3.12]). Therefore, "being isogenous" is an equivalence relation on the set of Drinfeld modules of rank r defined over F. Note that an isogeny $f : \phi \to \psi$ is an isomorphism if and only if $\deg_\tau(f) = 0$, see [Pap23, §3.8].

Let $f : \phi \to \psi$ be an isogeny of Drinfeld modules over F. Proposition 3.3.4(3) in [Pap23] implies that f is automatically separable *i.e.*, $f(x)$ has distinct roots in $\overline{F}$. We define, as we may, the kernel $\ker(f)$ of f as the set of zeroes of $f(x) \in F[x]$ in some algebraic closure over F (see Definition 1.1):

$$\ker(f) = \left\{\beta \in \overline{F} \mid f(\beta) = 0\right\} \subset \overline{F}.$$

Actually, since $f(x)$ is separable, we have $\ker(f) \subset F^{\mathrm{sep}}$. Then $\ker(f)$ is a finite dimensional $\mathbb{F}_q$-vector space contained in F^{sep}, with $\dim_{\mathbb{F}_q} \ker(f) = \deg_\tau(f)$. Furthermore, notice that $\ker(f)$ is stable under the action of ϕ_a for all $a \in A$: indeed, for any $\beta \in \ker(f)$ and any $a \in A$, we have

$$f(\phi_a(\beta)) = (f \cdot \phi_a)(\beta) = (\psi_a \cdot f)(\beta) = \psi_a(f(\beta)) = \psi_a(0) = 0.$$

Since $f \in F\{\tau\}$, the Galois action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on F^{sep} stabilizes $\ker(f)$.

This motivates the following definition:

Definition 1.7. Let ϕ be a Drinfeld module over F. Let $\mathtt{SG}_F(\phi)$ denote the set of finite dimensional $\mathbb{F}_q$-vector spaces G, contained in F^{sep}, which are both stable under the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ and stable under the action of ϕ_a for all $a \in A$ (*i.e.*, $\phi_a(x) \in G$ for all $x \in G$).

Note that, for any $G_1, G_2 \in \mathtt{SG}_F(\phi)$, both $G_1 \cap G_2$ and $G_1 + G_2$ are elements of $\mathtt{SG}_F(\phi)$.

Proposition 1.8. *Let ϕ be a Drinfeld module defined over F.*

(i) Let $f : \phi \to \psi$ be an isogeny from ϕ to a Drinfeld module ψ defined over F. Then $\ker(f) \in \mathtt{SG}_F(\phi)$.

(ii) For any $G \in \mathtt{SG}_F(\phi)$, there is a unique Drinfeld module ϕ/G defined over F and a unique normalized *isogeny $f_G : \phi \to \phi/G$ such that* $\ker(f_G) = G$.

For a given Drinfeld module ϕ, the elements $G \in \mathtt{SG}_F(\phi)$ therefore play a rôle similar to that of finite subgroup schemes of an abelian variety over F, in that they are both kernels of isogenies in their respective setting.

Proof. The first item follows from the above discussion, we only need to prove the second item. Given an element $G \in \mathtt{SG}_F(\phi)$ and a non-zero constant $c \in F^*$, we may define a polynomial $f_{G,c} \in F\{\tau\}$ as in the proof of Lemma 1.2.

Consider the polynomial $\varpi := f_{G,c} \cdot \phi_T \in F\{\tau\}$. For any $\alpha \in G$, we have $\phi_T(\alpha) \in G$ since we have assumed that G is stable under the action of ϕ_T. Then $\varpi(\alpha) = f_{G,c}(\phi_T(\alpha)) = 0$. In particular $G = \ker(f_{G,c}) \subseteq \ker(\varpi)$. The link between kernels and divisibility in $F\{\tau\}$ (Lemma 1.3) implies that there exists a unique $g_c \in F\{\tau\}$ such that $\varpi = g_c \cdot f_{G,c}$. The constant term of $f_{G,c}$ is non-zero, hence the constant term of g_c is T. We may now define a Drinfeld module $\psi_c : A \to F\{\tau\}$ by setting $\psi_c : T \mapsto g$. By construction, we have $f_{G,c} \cdot \phi_T = \psi_{c,T} \cdot f_{G,c}$ *i.e.*, $f_{G,c} \in F\{\tau\}$ is an isogeny $\phi \to \psi_c$.

Notice that, for any $c \in F^*$, we have $f_{G,c} = c\, f_{G,1}$ and $\psi_c = c\,\psi_1\, c^{-1} \simeq \psi_1$. Therefore, up to isomorphim, there is a unique pair (ψ, f) formed by a Drinfeld module ψ defined over F and an isogeny $f : \phi \to \psi$ with kernel $\ker(f) = G$. Indeed any isogeny $f \in F\{\tau\}$ with kernel G is of the form $f_{G,c}$ for some $c \in F^*$. Finally, there is one choice of $c \in F^*$ such that $f_{G,c}$ has constant coefficient 1, namely $c = \prod_{\beta \in G \smallsetminus \{0\}} (-\beta)^{-1}$. We set $\phi/G = \psi_c$ and $f_G = f_{G,c}$ for this choice of c. □

We also need a factorization property of isogenies:

Proposition 1.9. *Let ϕ be a Drinfeld module defined over F. Assume there are isogenies $f_1 : \phi \to \phi_1$ and $f_2 : \phi \to \phi_2$ defined over F such that* $\ker(f_1) \subset \ker(f_2)$. *Then there exists a unique isogeny $g : \phi_1 \to \phi_2$ such that $f_2 = g \cdot f_1$. Moreover, if f_1 and f_2 are normalized, then so is g.*

Proof. By Lemma 1.3, there exists a unique $g \in F\{\tau\}$ such that $f_2 = g \cdot f_1$. Using that $f_1 : \phi \to \phi_1$ and $f_2 : \phi \to \phi_2$ are isogenies we derive that, for any $a \in A$,

$$\begin{aligned}(g \cdot \phi_{1,a} - \phi_{2,a} \cdot g) \cdot f_1 &= g \cdot \phi_{1,a} \cdot f_1 - \phi_{2,a} \cdot g \cdot f_1 = g \cdot f_1 \cdot \phi_{1,a} - \phi_{2,a} \cdot f_2 \\ &= f_2 \cdot \phi_a - \phi_{2,a} \cdot f_2 = 0.\end{aligned}$$

The ring $F\{\tau\}$ has no zero-divisors and $f_1 \neq 0$, so we have $g \cdot \phi_{1,a} = \phi_{2,a} \cdot g$ for all $a \in A$ *i.e.*, g is an isogeny from ϕ_1 to ϕ_2. That g is normalized if f_1 and f_2 are is immediate. □

1.3. Stable reduction of Drinfeld modules. – Let K be a *finite* field extension of K_0. Consider a Drinfeld module $\phi : A \to K\{\tau\}$ of rank $r \geq 1$ defined over K.

Definition 1.10. For any place v of K, we say that ϕ *has stable reduction at* v if there exists a Drinfeld module $\Phi_v : A \to K\{\tau\}$ defined over K such that:

- Φ_v is K-isomorphic to ϕ, *i.e.*, there exists $c \in K^*$ such that $(\Phi_v)_T = c^{-1} \cdot \phi_T \cdot c$,
- and, writing $(\Phi_v)_T = T + \sum_{i=1}^{r} g_i \tau^i$ with $g_i \in K$, we have $\min\{\mathrm{ord}_v(g_i),\ 1 \leq i \leq r\} = 0$, *i.e.*, the coefficients of $(\Phi_v)_T$ are integral at v and at least one of them is a unit at v.

We say that ϕ has *stable reduction everywhere* if it has stable reduction at all finite places of K.

See [Pap23, §6.1] or [Gos12, §4.10] for detailed treatments of this notion. We note one important result about stable reduction (see [Hay79, Proposition 7.2] or [DD99, Lemme 2.10]):

Theorem 1.11. *Let K be a finite extension of K_0, and let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank $r \geq 1$ defined over K. Then there exists a finite extension K'/K such that the "base changed" Drinfeld module $\phi : A \to K'\{\tau\}$ has stable reduction everywhere.*

Isogenies of Drinfeld modules preserve stable reduction at a place of K. This is the content of Proposition 5.1 below (see also Proposition 2.4 in [Tag93]), which we defer to a later point in this paper because the proof of this statement requires tools developed in section 3.

2. A-lattices and Drinfeld modules

We let $\mathbb{C}_\infty$ denote the completion of an algebraic closure of the completion $K_{0,\infty}$ of K_0 at the place ∞. The field $\mathbb{C}_\infty$ is equipped with a non-archimedean valuation, extending $\mathrm{ord}_\infty : K_0^* \to \mathbb{Z}$. We let $|\cdot|_\infty$ denote the corresponding normalized absolute value, and we endow $\mathbb{C}_\infty$ with the induced topology.

2.1. A-lattices and their isogenies. – We recall the following definition (see [Gos12, Def. 4.1.2] or [Pap23, Def. 5.2.1]).

Definition 2.1. An *A-lattice* (or simply a *lattice*) is a finitely generated A-submodule Λ of $\mathbb{C}_\infty$, which is discrete as a subset of $\mathbb{C}_\infty$ (*i.e.*, Λ has finite intersection with any ball in $\mathbb{C}_\infty$).

A lattice Λ is necessarily torsion-free; the *rank* of Λ is its rank as a finitely generated A-module.

Let Λ be an A-lattice of rank $r \geq 1$. Since $A = \mathbb{F}_q[T]$ is a principal domain and since $\Lambda \subset \mathbb{C}_\infty$ is discrete, there exist $\omega_1, \omega_2, \ldots, \omega_r \in \mathbb{C}_\infty$ such that $\Lambda = A\,\omega_1 + A\,\omega_2 + \cdots + A\,\omega_r$ and the ω_i are $K_{0,\infty}$-linearly independent (see [Pap23, Proposition 5.3.8]).

Definition 2.2. Let $\Lambda_1, \Lambda_2 \subset \mathbb{C}_\infty$ be two A-lattices *of the same rank* $r \geq 1$. A morphism $c : \Lambda_1 \to \Lambda_2$ is an element $c \in \mathbb{C}_\infty$ such that $c\,\Lambda_1 \subset \Lambda_2$. Such a morphism is called an *isogeny* if $c \neq 0$.

If there exists an isogeny between two lattices, they will be called *isogenous*.

Two lattices Λ_1 and Λ_2 are isomorphic if and only if there is a $c \in \mathbb{C}_\infty \smallsetminus \{0\}$ such that $c\,\Lambda_1 = \Lambda_2$.

For any isogeny $c : \Lambda_1 \to \Lambda_2$, the A-lattice $c\,\Lambda_1$ has finite index in Λ_2 *i.e.*, the quotient A-module $\Lambda_2/c\,\Lambda_1$ is a finite A-module. Since $A = \mathbb{F}_q[T]$ is a principal domain, the structure theorem for torsion A-modules (see [Lang, III.7, Thm. 7.7]) implies that $\Lambda_2/c\,\Lambda_1$ is a direct sum of finitely many A-modules of the form $A/a_i\,A$ (with non-constant $a_i \in A$). In particular, the quotient $\Lambda_2/c\,\Lambda_1$ is a finite dimensional $\mathbb{F}_q$-vector space and thus a finite set: we usually denote its cardinality $|\Lambda_2/c\,\Lambda_1|$ (a power of q) as an index $(\Lambda_2 : c\,\Lambda_1)$.

We refer to Theorem 5.3.10 of [Pap23] for many more details about A-lattices and isogenies of A-lattices.

2.2. Covolume of A-lattices. – Let Λ be an A-lattice of rank $r \geq 1$. A *successive minimum basis* of Λ is an ordered basis $(\eta_1, \ldots, \eta_r)$ of Λ as an A-module such that $|\eta_1|_\infty \geq |\eta_2|_\infty \geq \cdots \geq |\eta_r|_\infty$, and, for any $k = 1, \ldots, r$,

$$|\eta_k|_\infty = \min\left(\left\{\left|\lambda - \sum_{i=k+1}^{r} a_i\, \eta_i\right|_\infty,\ (a_{k+1}, \ldots, a_r) \in A^{r-k+1},\ \lambda \in \Lambda\right\} \smallsetminus \{0\}\right).$$

In other words, η_r has minimal absolute value among non-zero elements of Λ, and each η_k has minimal absolute value among elements of $\Lambda \smallsetminus (A\,\eta_{k+1} + \cdots + A\,\eta_r)$. See [BPR21, §4.1] or [Pap23, §5.5].

By the discussion in [Pap23, §5.5], any A-lattice Λ of positive rank admits a successive minimum basis; such a basis is not necessarily unique, but the tuple $(|\eta_1|_\infty, \ldots, |\eta_r|_\infty)$ is an invariant of Λ. We then define the *covolume of* Λ by

$$\mathcal{D}(\Lambda) \coloneqq |\eta_1|_\infty\, |\eta_2|_\infty \cdots |\eta_r|_\infty, \tag{2.1}$$

for any successive minimum basis $(\eta_1, \ldots, \eta_r)$ of Λ. By the previous paragraph, $\mathcal{D}(\Lambda)$ is well-defined *i.e.*, independent of the choice of a successive minimum basis of Λ.

It will be relevant for us that the covolume interacts well with isogenies of A-lattices:

Lemma 2.3. *Let $c \in \mathbb{C}_\infty^*$ be an isogeny $c : \Lambda \to \Lambda'$ between two A-lattices Λ, Λ' of rank $r \geq 1$. Then we have*

$$\mathcal{D}(\Lambda') = |c|_\infty^r\, (\Lambda' : c\,\Lambda)\, \mathcal{D}(\Lambda).$$

In particular, if Λ and Λ' are two A-lattices of rank $r \geq 1$ such that $\Lambda \subseteq \Lambda'$, then

$$\mathcal{D}(\Lambda') = (\Lambda' : \Lambda)\, \mathcal{D}(\Lambda).$$

Proof. This follows directly from Lemma 4.1 in [BPR21]. □

2.3. Drinfeld modules and lattices. – We now recall the link between the theory of A-lattices and that of Drinfeld modules over $\mathbb{C}_\infty$.

Theorem 2.4. *For any $r \geq 1$, there is an equivalence of categories between*

(a) the category whose objects are the Drinfeld modules $\phi : A \to \mathbb{C}_\infty\{\tau\}$ of rank r defined over $\mathbb{C}_\infty$, and whose morphisms are the morphisms of Drinfeld modules as in Definition 1.6,

(b) the category whose objects are A-lattices $\Lambda \subset \mathbb{C}_\infty$ of rank r, and whose morphisms are the morphisms of lattices as in Definition 2.1.

This is Theorem 4.6.9 in [Gos12]. The correspondence can be made more explicit by using the exponential function, as we now briefly explain. To any A-lattice $\Lambda \subset \mathbb{C}_\infty$, one associates a surjective Λ-periodic entire function $\mathrm{e}_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by a convergent product

$$\forall z \in \mathbb{C}_\infty, \qquad \mathrm{e}_\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' \left(1 - \frac{z}{\lambda}\right),$$

where $\prod'$ means that the product is taken over all non-zero elements of Λ. We refer to [Gos12, §4.2] or [Pap23, §5.1] for details. For any $a \in A$, let

$$\phi_a^\Lambda(x) \coloneqq \frac{1}{a}\, \mathrm{e}_\Lambda(a\, \mathrm{e}_\Lambda^{-1}(x)),$$

where $\mathrm{e}_\Lambda^{-1}(x)$ denotes an element $z \in \mathbb{C}_\infty$ such that $\mathrm{e}_\Lambda(z) = x$. One shows that $\phi_a^\Lambda(x)$ is actually a $\mathbb{F}_q$-linear polynomial in $\mathbb{C}[x]$, hence comes from an element $\phi_a^\Lambda \in \mathbb{C}_\infty\{\tau\}$. One then proves that the map, $\phi^\Lambda : A \to \mathbb{C}_\infty\{\tau\}$ given by $a \mapsto \phi_a^\Lambda$ defines a Drinfeld module of rank r. The next step is to show that any Drinfeld module of rank r over $\mathbb{C}_\infty$ arises in this way. We refer to [Gos12, §4.3] or [Pap23, §5.2] for more details and proofs.

On the level of morphisms, the correspondence is given as follows. Let ϕ, ψ be two Drinfeld modules defined over $\mathbb{C}_\infty$. Write $\Lambda_\phi, \Lambda_\psi$ for the A-lattices associated to ϕ and ψ by the previous paragraph. Then the map

$$\begin{array}{ccc} \{\text{morphisms of Drinfeld modules } \phi \to \psi\} & \longrightarrow & \{\text{morphisms of } A\text{-lattices } \Lambda_\phi \to \Lambda_\psi\} \\ f \in \mathbb{C}_\infty\{\tau\} & \mapsto & \text{constant coefficient of } f \end{array}$$

is the desired bijection. The reader is invited to consult [Gos12, §4.6] or [Pap23, §5.2] for more details. For later use, we record the following result, which is essentially Theorem 5.2.10 in [Pap23]:

Proposition 2.5. *Let $f : \phi \to \psi$ be an isogeny of Drinfeld modules defined over $\mathbb{C}_\infty$. Write $f_0 \in \mathbb{C}_\infty$ for the constant coefficient of f. Let Λ_ϕ and Λ_ψ be the A-lattices associated to ϕ and ψ by Theorem 2.4, respectively. The isogeny $c : \Lambda_\phi \to \Lambda_\psi$ of lattices induced by f is $c = f_0$. Moreover, there is an isomorphism of $\mathbb{F}_q$-vector spaces $\ker(f) \simeq \Lambda_\psi / f_0 \Lambda_\phi$ and we have $\deg_\tau f = \log_q (\Lambda_\psi : f_0 \Lambda_\phi)$.*

In particular, a *normalized* isogeny $f : \phi \to \psi$ induces an *inclusion* of A-lattices $\Lambda_\phi \subseteq \Lambda_\psi$, and

$$\deg_\tau f = \dim_{\mathbb{F}_q} \ker(f) = \dim_{\mathbb{F}_q} \Lambda_\psi / \Lambda_\phi = \log_q (\Lambda_\psi : \Lambda_\phi).$$

2.4. A covolume identity. – We now prove the following:

Theorem 2.6. *Let $\Lambda, \Lambda_1, \Lambda_2$ be three A-lattices of the same rank $r \geq 1$ such that $\Lambda \subseteq \Lambda_1 \cap \Lambda_2$. Then $\Lambda_1 \cap \Lambda_2$ and $\Lambda_1 + \Lambda_2$ are also A-lattices of rank r. Moreover, we have*

$$\log_q \mathcal{D}(\Lambda_1 \cap \Lambda_2) + \log_q \mathcal{D}(\Lambda_1 + \Lambda_2) = \log_q \mathcal{D}(\Lambda_1) + \log_q \mathcal{D}(\Lambda_2). \tag{2.2}$$

Identity (2.2) may be thought of as a "parallelogram equality" for A-lattices. The data of the statement indeed naturally gives rise to a diagram of A-lattices where arrows represent inclusions:

$$\begin{array}{ccccc} & & \Lambda_1 \cap \Lambda_2 & & \\ & \swarrow & & \searrow & \\ \Lambda_1 & & & & \Lambda_2\,. \\ & \searrow & & \swarrow & \\ & & \Lambda_1 + \Lambda_2 & & \end{array}$$

Proof. Recall that $A = \mathbb{F}_q[T]$ is a principal domain. As a sub-A-module of the finitely generated free A-module Λ_1 of rank r, $\Lambda_1 \cap \Lambda_2$ is also a free A-module of rank $\leq r$. On the other hand, the A-module Λ of rank r is contained in $\Lambda_1 \cap \Lambda_2$. The latter must therefore have rank exactly r. In particular, $\Lambda_1 \cap \Lambda_2$ has finite index in Λ_2. The sum $\Lambda_1 + \Lambda_2$ is a sub-A-module of the free A-module $\mathbb{C}_\infty$, and it contains the A-module Λ_1 of rank r. Hence $\Lambda_1 + \Lambda_2$ is a free finitely generated A-module of rank $\geq r$. The "second isomorphism theorem" for A-modules (see p. 120 in [Lang, §III.1]) shows that there is an isomorphism of A-modules

$$(\Lambda_1 + \Lambda_2)/\Lambda_1 \simeq \Lambda_2/(\Lambda_1 \cap \Lambda_2). \tag{2.3}$$

In particular, the quotient $(\Lambda_1 + \Lambda_2)/\Lambda_1$ is a finite A-module, and $\Lambda_1 + \Lambda_2$ has rank equal to r. The fact that both $\Lambda_1 \cap \Lambda_2$ and $\Lambda_1 + \Lambda_2$ are discrete subsets of $\mathbb{C}_\infty$ follows easily from the discreteness of Λ_1, Λ_2 in $\mathbb{C}_\infty$. The first assertion is proved.

Combined with (2.3), the "third isomorphism theorem" for A-modules (*loc. cit.*) yields A-module isomorphisms

$$\frac{(\Lambda_1 + \Lambda_2)/\Lambda}{\Lambda_1/\Lambda} \simeq \frac{\Lambda_1 + \Lambda_2}{\Lambda_1} \simeq \frac{\Lambda_2}{\Lambda_1 \cap \Lambda_2} \simeq \frac{\Lambda_2/\Lambda}{(\Lambda_1 \cap \Lambda_2)/\Lambda}.$$

In particular, we obtain $(\Lambda_1 + \Lambda_2 : \Lambda)\,(\Lambda_1 \cap \Lambda_2 : \Lambda) = (\Lambda_1 : \Lambda)\,(\Lambda_2 : \Lambda)$. Multiplying on both sides by $\mathcal{D}(\Lambda)^2$ and using the second identity in Lemma 2.3, we get

$$\mathcal{D}(\Lambda_1 + \Lambda_2)\,\mathcal{D}(\Lambda_1 \cap \Lambda_2) = \mathcal{D}(\Lambda_1)\,\mathcal{D}(\Lambda_2).$$

Taking the image by the logarithm $\log_q$ yields the desired equality. ☐

3. Valuation polygons of twisted polynomials

For the duration of this section, we work in the following setting: let F be a field containing $\mathbb{F}_q$, equipped with a discrete non-archimedean valuation $\nu : F \to \mathbb{Z} \cup \{\infty\}$, normalized so that $\nu(F^*) = \mathbb{Z}$. We fix an algebraic closure $\overline{F}$ of F and choose an extension of ν to $\overline{F}$, also denoted by $\nu : \overline{F} \to \mathbb{Q} \cup \{\infty\}$.

Definitions and basic properties of the objects described in this section can essentially be found in [Tag91, §1] and [Tag93, §2]. We thought it worthwhile to rewrite detailed definitions and some of the proofs in our notation, in order to keep this paper self-contained and clarify some steps.

3.1. Valuation polygons. – Let $f = \sum_{i=0}^{d} f_i \tau^i \in F\{\tau\}$ be a non-zero polynomial. For $z \in \mathbb{R}$, let

$$V_f(z) \coloneqq \min\{\nu(f_i) + q^i z,\ 0 \le i \le d\}. \tag{3.1}$$

The map thus defined is continuous and (strictly) increasing on $\mathbb{R}$, and $V_f : \mathbb{R} \to \mathbb{R}$ is a bijection. By definition, its graph is concave broken line with finitely many slopes, each of which is an element of $\{1, q, \dots, q^d\}$. One checks that the abscissae of the break-points are exactly the elements of

$$\mathcal{E}_f \coloneqq \{\nu(\beta),\ \beta \in \ker(f) \smallsetminus \{0\}\} \subset \mathbb{Q}.$$

We also note another way of characterizing V_f:

Lemma 3.1. *For all $x \in \overline{F}$, we have $\nu(f(x)) \ge V_f(\nu(x))$. Furthermore, for all $x \in \overline{F}$ such that $\nu(x) \notin \mathcal{E}_f$, we have $\nu(f(x)) = V_f(\nu(x))$.*

Proof. The first statement is a direct consequence of the non-archimedean triangle inequality (see [Pap23, §2.1] for instance). A direct computation shows that

$$\mathcal{E}_f \coloneqq \left\{-\frac{\nu(f_j) - \nu(f_i)}{q^j - q^i},\ 0 \le i \ne j \le d\right\}.$$

For any $x \in \overline{F}$ with $\nu(x) \notin \mathcal{E}_f$, the $d+1$ numbers $\nu(f_i) + q^i \nu(x) = \nu(f_i x^{q^i})$, $0 \le i \le d$, are pairwise distinct. The second assertion now follows from the first one, and the case of equality in the triangle inequality. □

The graph of V_f is sometimes called the valuation polygon of f, and it is in a sense "dual" to the Newton polygon of $f(x)/x \in F[x]$, as explained in [Rob85, §2.1–§2.2]. Write the elements of $\mathcal{E}_f$ in increasing order: $e_1 < e_2 < \dots < e_m$. It will be convenient for later to rename e_m:

$$\lambda_\nu(f) \coloneqq \max\{\nu(\beta),\ \beta \in \ker(f) \smallsetminus \{0\}\} = -\min\left\{\frac{\nu(f_i) - \nu(f_0)}{q^i - 1},\ 1 \le i \le d\right\}. \tag{3.2}$$

It is clear that $V_f(z) \le z + \nu(f_0)$ for all $z \in \mathbb{R}$. Notice (as in [Tag91, Remark 1.4]) that

$$\lambda_\nu(f) = \min\{z \in \mathbb{R} \mid V_f(z) = z + \nu(f_0)\}. \tag{3.3}$$

For any $z \in \mathbb{R}$, set

$$N_f(z) \coloneqq |\{\beta \in \ker(f) \mid \nu(\beta) \ge z\}|.$$

The map $z \mapsto N_f(z)$ is non-increasing, left-continuous and piecewise constant. Since $f(x)$ is a $\mathbb{F}_q$-linear polynomial and ν is non-archimedean, the set $\{\beta \in \ker(f) \mid \nu(\beta) \ge z\}$ is a finite dimensional $\mathbb{F}_q$-vector space, so that N_f has values in $\{1, q, \dots, q^d\}$. By definition, the graph of N_f has jumps at elements of $\mathcal{E}_f$ and is constant on each segment $(e_k, e_{k+1}]$, as well as on $(-\infty, e_1]$ and on $(e_m, +\infty)$. For $z \le e_1$, it is clear that $N_f(z) = |\ker(f)| = q^d$. For $z > e_m$, we have $N_f(z) = 1$ (the only root of $f(x)$ with valuation larger than e_m is 0). For $1 \le k < m$ and $z \in (e_k, e_{k+1}]$, one readily checks (by an argument similar to the proof of Theorem 2.5.2 in [Pap23] for instance, see [Rob85, §2.1]) that $V_f(z) = \nu(f_i) + N_f(z)\, z$, *i.e.*, that the index $i \in \{1, \ldots, r\}$ realizing the minimum in $V_f(z)$ is $\log_q N_f(z)$. We conclude that the maps V_f and N_f are related as follows:

$$V_f(z) = \begin{cases} \nu(f_d) + q^d\, z & \text{if } z < e_1, \\ \nu(f_i) + q^i\, z & \text{if } z \in (e_k, e_{k+1}] \text{ for } 1 \le k < m, \text{ where } i = \log_q N_f(z), \\ \nu(f_0) + q^0\, z & \text{if } z > e_m = \lambda_\nu(f). \end{cases}$$

In other words, we have

Lemma 3.2. *For any $z \in \mathbb{R} \smallsetminus \mathcal{E}_f$, the map V_f is differentiable at z, and ${V_f}'(z) = N_f(z)$.*

3.2. An inequality. – We now prove the main result of this section, which relates the valuation polygons of two polynomials to those of their GCD and LCM.

Theorem 3.3. *Let $f, g \in F\{\tau\}$ be normalized polynomials. Let d and ℓ, denote their normalized GCD and LCM in $F\{\tau\}$, respectively. Then, for all $z \in \mathbb{R}$,*

$$V_d(z) + V_\ell(z) \le V_f(z) + V_g(z).$$

In the special case $d = 1$ (*i.e.*, when f and g have trivial GCD in $F\{\tau\}$) this inequality becomes $V_\ell(z) \le V_f(z) + V_g(z) - z$ for all $z \in \mathbb{R}$. Notice indeed that $V_1(z) = z$ for all $z \in \mathbb{R}$.

Proof. We begin by proving a lemma:

Lemma 3.4. *In the context of the theorem, we have*

$$\forall z \in \mathbb{R}, \qquad N_d(z)\, N_\ell(z) \ge N_f(z)\, N_g(z) \qquad \textit{and} \qquad N_d(z) + N_\ell(z) \ge N_f(z) + N_g(z).$$

Proof (of Lemma 3.4). For $h \in \{f, g, d, \ell\}$, we write $S_h(z) := \{\beta \in \ker(h) \mid \nu(\beta) \ge z\}$. Notice that $S_h(z)$ is a $\mathbb{F}_q$-vector space contained in $\ker(h)$, and that $|S_h(z)| = N_h(z)$. Recall from Lemma 1.4 that $\ker(\ell) = \ker(f) + \ker(g)$ and $\ker(d) = \ker(f) \cap \ker(g)$. For a fixed $z \in \mathbb{R}$, consider the map

$$\sigma : S_f(z) \times S_g(z) \to S_\ell(z), \quad (\beta, \beta') \mapsto \beta + \beta'.$$

This defines a $\mathbb{F}_q$-linear map because $\ker(f) + \ker(g) = \ker(\ell)$ and, for any β, β' such that $\nu(\beta) \ge z$, $\nu(\beta') \ge z$, we have $\nu(\beta + \beta') \ge \min\{\nu(\beta), \nu(\beta')\} \ge z$. The kernel of σ is a diagonally embedded copy of $S_f(z) \cap S_g(z) = S_d(z)$ in $S_f(z) \times S_g(z)$. Hence σ induces an injective $\mathbb{F}_q$-linear map

$$(S_f(z) \times S_g(z))/S_d(z) \longrightarrow S_\ell(z).$$

Comparing cardinalities of the domain and codomain of this latter map yields the first inequality.

To prove the second inequality, note that $1 \le N_d(z) \le N_f(z), N_g(z) \le N_\ell(z)$ for all $z \in \mathbb{R}$; thus

$$\begin{aligned} N_d(z)\, N_\ell(z) &\ge N_f(z)\, N_g(z) \\ &\ge (N_f(z) - N_d(z))(N_g(z) - N_d(z)) + N_f(z)\, N_d(z) + N_g(z)\, N_d(z) - N_d(z)^2 \\ &\ge N_f(z)\, N_d(z) + N_g(z)\, N_d(z) - N_d(z)^2. \end{aligned}$$

The result follows by dividing by $N_d(z) \ge 1$. □

Pick z_0 large enough so that $z_0 > z$ and so that $V_d(z_0) = V_\ell(z_0) = V_f(z_0) = V_g(z_0) = z_0$ (we may do so because f, g, d, ℓ are normalized). For $h \in \{f, g, d, \ell\}$, recall that V_h is continuous on $\mathbb{R}$, is differentiable almost everywhere (the exceptional abscissae are the elements of $\mathcal{E}_h$), and that ${V_h}'(z) = N_h(z)$ for any $z \in \mathbb{R} \smallsetminus \mathcal{E}_h$. We thus have

$$V_h(z) = z_0 - \int_z^{z_0} N_h(t)\, \mathrm{d}t.$$

Using that $z < z_0$, we now deduce from the above lemma that

$$\begin{aligned} V_d(z) + V_\ell(z) = 2\, z_0 - \int_z^{z_0} (N_d(t) + N_\ell(t))\ \mathrm{d}t &\le 2\, z_0 - \int_z^{z_0} (N_f(t) + N_g(t))\ \mathrm{d}t \\ = 2\, z_0 - V_f(z_0) + V_f(z) - V_g(z_0) + V_g(z) &= V_f(z) + V_g(z). \end{aligned}$$

The desired inequality is proved. □

3.3. Two more properties of valuation polygons. – In order to use the results of this section in the context of Drinfeld modules (see §5.4 below), we require the following two properties (which are essentially [Tag91, Lemma 1.2] and [Tag93, Lemma 2.3], respectively).

Lemma 3.5. *Let $f, g \in F\{\tau\}$ be non-zero polynomials. For all $z \in \mathbb{R}$, we have*

$$V_{f\cdot g}(z) = V_f \circ V_g(z).$$

Proof. By Lemma 3.1, for almost all $z \in \mathbb{Q}$ and any $x \in \overline{F}$ with $\nu(x) = z$, we have

$$V_{f\cdot g}(z) = V_{f\cdot g}(\nu(x)) = \nu((f\cdot g)(x)) = \nu(f(g(x))) = V_f(\nu(g(x))) = V_f(V_g(z)).$$

The two maps $z \mapsto V_{f\cdot g}(z)$ and $z \mapsto V_f \circ V_g(z)$ are continuous on $\mathbb{R}$. We have just noted that they coincide on a dense subset of $\mathbb{R}$, so they are equal. □

Proposition 3.6. *Let $\phi, \psi \in F\{\tau\}$ be two polynomials such that $\nu(\phi_0) = \nu(\psi_0) = 0$. For any non-zero polynomial $f \in F\{\tau\}$ such that $f \cdot \phi = \psi \cdot f$, we have*

$$\lambda_\nu(\psi) = V_f(\lambda_\nu(\phi)).$$

Proof. If $\varphi \in F\{\tau\}$ satisfies $\nu(\varphi_0) = 0$, the discussion in §3.1 shows that $V_\varphi(z) \le z$ for all $z \in \mathbb{R}$, and $\lambda_\nu(\varphi) = \min\{z \in \mathbb{R} \mid V_\varphi(z) = z\}$. In the context of the proposition, let $\lambda' = V_f(\lambda_\nu(\phi))$.

It follows from Lemma 3.5 that $V_\psi = V_f \circ V_\phi \circ V_f^{-1}$ and therefore $V_\psi(\lambda') = \lambda'$. The preceding remark implies that $\lambda' \ge \lambda_\nu(\psi)$. Recall that V_f, V_ϕ, and V_ψ are strictly increasing bijections, and note that V_f^{-1} is also strictly increasing. For any $z < \lambda'$, we thus have

$$V_\psi(z) < V_\psi(\lambda') = V_f \circ V_\phi \circ V_f^{-1}(\lambda') = V_f(\lambda_\nu(\phi)) = \lambda' < z.$$

This shows that $\lambda' \le \lambda_\nu(\psi)$. We conclude that $\lambda_\nu(\psi) = \lambda' = V_f(\lambda_\nu(\phi))$. □

4. Heights of Drinfeld modules

We now define two notions of heights for Drinfeld modules defined over a finite extension K of K_0.

4.1. Graded height. – Let ϕ be a Drinfeld $\mathbb{F}_q[T]$-module of rank $r \ge 1$ defined over K. Then ϕ is characterized by the twisted polynomial

$$\phi_T = T + g_1\,\tau + g_2\,\tau^2 + \cdots + g_r\,\tau^r \in K\{\tau\}, \qquad g_i \in K,\ g_r \neq 0.$$

Let v be a place of K. We define the *local component at v of the graded height* by

$$h^v_{\mathrm{Gr},K}(\phi) \coloneqq -\min\left\{\frac{\mathrm{ord}_v(g_i)}{q^i - 1}\,,\ 1 \le i \le r\right\} \in \mathbb{Q}. \tag{4.1}$$

Let F denote the completion of K at v, equipped with the (normalized) valuation $\nu = \mathrm{ord}_v$, notice that $h^v_{\mathrm{Gr},K}(\phi)$ is nothing but the maximal break-point $\lambda_\nu(\phi_T)$ of the valuation polygon of $\phi_T \in F\{\tau\}$ defined in (3.2) in the previous section. We will also write $\lambda_v(\phi)$ for $h^v_{\mathrm{Gr},K}(\phi)$.

We record the following statement (see [Pap23, Lemma 6.1.5(1)] or [Tag93, p. 301]):

Proposition 4.1. *Let ϕ be as above, and v be a place of K. Then ϕ has stable reduction at v if and only if $h^v_{\mathrm{Gr},K}(\phi) = \lambda_v(\phi)$ is an integer.*

Proof. A Drinfeld module $\psi : A \to K\{\tau\}$ defined over K is isomorphic to ϕ if and only if there exists $c \in K^*$ such that $\psi = c^{-1} \cdot \phi \cdot c$. A straightforward computation shows that $\lambda_v(\psi) = \lambda_v(\phi) - \mathrm{ord}_v(c)$. By definition ϕ has stable reduction at v if and only if there exists $c \in K^*$ such that $\lambda_v(c^{-1}\cdot\phi\cdot c) = 0$ *i.e.*, such that $\mathrm{ord}_v(c) = \lambda_v(\phi)$. This happens exactly when $\lambda_v(\phi) \in \mathrm{ord}_v(K^*) = \mathbb{Z}$. □

Note that $h^v_{\mathrm{Gr},K}(\phi) = 0$ for all but finitely many places $v \in M_K$. We can then define the *graded height of* ϕ by

$$h_{\mathrm{Gr},K}(\phi) \coloneqq \frac{1}{[K : K_0]} \sum_{v \in M_K} f_v \, h^v_{\mathrm{Gr},K}(\phi). \tag{4.2}$$

One shows that, if ϕ' is a Drinfeld module over K which is isomorphic to ϕ, then $h_{\mathrm{Gr},K}(\phi) = h_{\mathrm{Gr},K}(\phi')$. We refer to [BPR21, §2.2] for a proof, as well as more details about the graded height.

One can describe the variation of local graded height by isogeny in terms of the valuation polygon of the isogeny:

Proposition 4.2. *Let v be a place of K. Let ϕ, ψ be two isogenous Drinfeld modules defined over K. Let $f : \phi \to \psi$ be a normalized isogeny. Then*

$$h^v_{\mathrm{Gr},K}(\psi) - h^v_{\mathrm{Gr},K}(\phi) = V_f(\lambda_v(\phi)) - \lambda_v(\phi).$$

Proof. Apply Proposition 3.6 and the previous discussion. □

4.2. Taguchi height. – Let ϕ be a Drinfeld module of rank $r \geq 1$ defined over K. For an infinite place $v \in M^\infty_K$, consider the corresponding field inclusion $\sigma_v : K \hookrightarrow K_v \hookrightarrow \mathbb{C}_\infty$. Let $\sigma_v \circ \phi : A \to \mathbb{C}_\infty\{\tau\}$ denote the "base changed" Drinfeld module. By Theorem 2.4 in section 2.3, one can unequivocally associate to the Drinfeld module $\sigma_v \circ \phi$ over $\mathbb{C}_\infty$ an A-lattice $\Lambda_{\phi,v} \subset \mathbb{C}_\infty$. Define the *local component at v of the Taguchi height* by

$$h^v_{\mathrm{Tag},K}(\phi) \coloneqq \log_q \left(\mathcal{D}(\Lambda_{\phi,v})^{-1/r} \right).$$

For a finite place $v \in M^f_K$, define the *local component at v of the Taguchi height* to be

$$h^v_{\mathrm{Tag},K}(\phi) \coloneqq \left\lceil h^v_{\mathrm{Gr},K}(\phi) \right\rceil,$$

where $\lceil\cdot\rceil$ denotes the ceiling function and $h^v_{\mathrm{Gr},K}(\phi)$ is given by (4.1).

We then define the *Taguchi height of* ϕ in the following manner:

$$h_{\mathrm{Tag},K}(\phi/K) \coloneqq \frac{1}{[K : K_0]} \left(\sum_{v \in M^f_K} f_v \, h^v_{\mathrm{Tag},K}(\phi) + \sum_{v \in M^\infty_K} n_v \, h^v_{\mathrm{Tag},K}(\phi) \right). \tag{4.3}$$

The sum in (4.3) is actually finite because $h^v_{\mathrm{Tag},K}(\phi) = 0$ for all but a finite number of places $v \in M_K$. This notion of height was introduced by Taguchi in [Tag91, Tag93]. We refer to [Wei17, Definition 4.1], [Wei20, Definition 5.1], and [BPR21, §4.3] for more in-depth accounts.

The Taguchi height is sometimes also called the differential height: one can indeed also define $h_{\mathrm{Tag},K}(\phi)$ as the degree of a certain line bundle related to differential forms "on" ϕ (as in [Tag93, §5.3] or [DD99, Définition 2.8]). This latter definition draws a clear parallel with the setting of abelian varieties over K, as studied in [GLP25].

In what follows, we only consider Drinfeld modules ϕ with stable reduction everywhere: in that case, by Proposition 4.1, one can rewrite (4.3) as

$$h_{\mathrm{Tag},K}(\phi/K) = \frac{1}{[K : K_0]} \left(\sum_{v \in M^f_K} f_v \, h^v_{\mathrm{Gr},K}(\phi) + \sum_{v \in M^\infty_K} n_v \, \log_q \left[\mathcal{D}(\Lambda_{\phi,v})^{-1/r} \right] \right). \tag{4.4}$$

5. Parallelogram inequalities

In this section, we gather the results in earlier sections to conclude the proofs of our main theorems: the different cases of Theorem B and, consequently, Theorem A.

5.1. Stable reduction and isogenies. − In this subsection, we show that isogenies preserve stable reduction at a place $v \in M_K$. This statement is a general version of [Tag93, Proposition 2.4].

Proposition 5.1. *Let K be a finite extension of K_0, and let ϕ, ψ be isogenous Drinfeld module defined over K. For any place v of K, one has:*

$$\phi \text{ has stable reduction at } v \text{ if only if } \psi \text{ has stable reduction at } v.$$

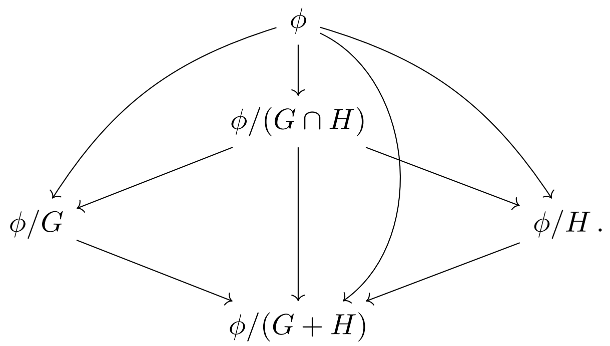
Proof. Being isogenous is an equivalence relation and, in particular, is symmetric (because of the existence of dual isogenies, [Pap23, Proposition 3.3.12]). It therefore suffices to prove only one of the implications.

Let $f : \phi \to \psi$ be an isogeny defined over K. Assume that ϕ has stable reduction at v. By Proposition 4.1 above, ψ has stable reduction at v if and only if $h^v_{\mathrm{Gr},K}(\psi) = \lambda_v(\psi)$ is an integer. We know from Proposition 3.6 that $\lambda_v(\psi) = V_f(\lambda_v(\phi))$, where $\lambda_v(\phi) \in \mathbb{Z}$ because ϕ has stable reduction at v. The definition (see (3.1)) of $V_f : \mathbb{R} \to \mathbb{R}$, and the fact that ord_v is normalized by $\mathrm{ord}_v(K^*) = \mathbb{Z}$, directly imply that $V_f(\mathbb{Z}) \subseteq \mathbb{Z}$. We conclude, as required, that $\lambda_v(\psi)$ is an integer and, by consequence, that ψ has stable reduction at v. □

5.2. Parallelogram configurations. − Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank $r \geq 1$ defined over a finite extension K of K_0, and let $G, H \in \mathtt{SG}_K(\phi)$. As was already noted in §1.2, both $G \cap H$ and $G + H$ are also elements of $\mathtt{SG}_K(\phi)$. Proposition 1.8 implies the existence and uniqueness of normalized isogenies in $K\{\tau\}$:

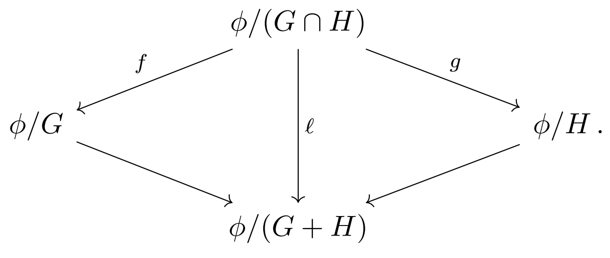
$$\begin{aligned} &\phi \to \phi/G \ \text{ with kernel } G, && && \phi \to \phi/H \ \text{ with kernel } H, \\ &\phi \to \phi/(G \cap H) \ \text{ with kernel } G \cap H, && \text{and} && \phi \to \phi/(G+H) \ \text{ with kernel } G+H. \end{aligned}$$

From the inclusions $G \cap H \subset G, H$ and $G, H \subset G + H$, and from Proposition 1.9, we can write $\phi \to \phi/G$ as the composition $\phi \to \phi/(G \cap H) \to \phi/G$ of two normalized isogenies. Similarly, we can write $\phi \to \phi/H$ as the composition $\phi \to \phi/(G \cap H) \to \phi/H$, etc. The situation is best summarized by the following commutative diagram in which arrows represent normalized isogenies:



(5.1)

This leads to a parallelogram of normalized isogenies:



In this parallelogram, note that $\ker(f) \cap \ker(g)$ is trivial: this follows from Lemma 1.4 because the isogeny $\phi \to \phi/(G \cap H)$ in diagram (5.1) is the GCD in $K\{\tau\}$ of $\phi \to \phi/G$ and $\phi \to \phi/H$. Thus, the GCD of f and g in $K\{\tau\}$ is 1, and their LCM in $K\{\tau\}$ is the isogeny $\ell : \phi/(G \cap H) \to \phi/(G + H)$.

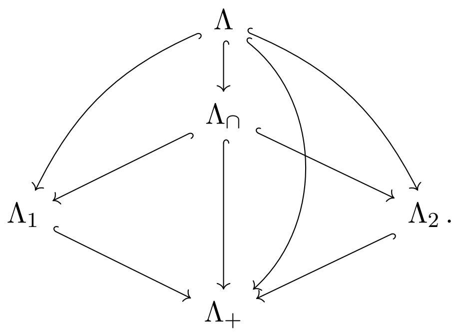
If v is a place of K and if ϕ has stable reduction at v, Proposition 5.1 ensures that the five Drinfeld modules appearing in diagram (5.1), which are all isogenous to ϕ, also have stable reduction at v.

5.3. Taguchi height at infinite places. – In this section, we prove:

Theorem 5.2. *Let ϕ be a Drinfeld module defined over K, and let $G, H \in \mathtt{SG}_K(\phi)$. For any infinite place $v \in M_K^\infty$ of K, we have*

$$h^v_{\mathrm{Tag},K}(\phi/(G\cap H)) + h^v_{\mathrm{Tag},K}(\phi/(G+H)) = h^v_{\mathrm{Tag},K}(\phi/G) + h^v_{\mathrm{Tag},K}(\phi/H). \tag{5.2}$$

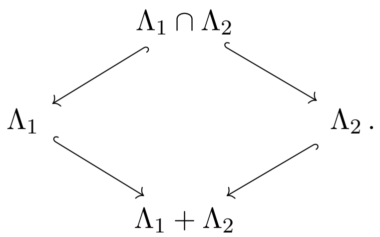
Proof. Let v be an infinite place of K and $\sigma_v : K \hookrightarrow K_v \hookrightarrow \mathbb{C}_\infty$ be the corresponding inclusion of fields. We "base change" all the Drinfeld modules (defined over K) involved in diagram (5.1) into Drinfeld modules over $\mathbb{C}_\infty$ *via* σ_v. We still denote them by the same letters. By the discussion in §2.3, there are A-lattices Λ, Λ_1, Λ_2, $\Lambda_\cap$ and Λ_+ of the same rank r corresponding to each of ϕ, ϕ/G, ϕ/H, $\phi/(G\cap H)$ and $\phi/(G+H)$, respectively. The isogenies in diagram (5.1) are all normalized. The correspondence between isogenies of Drinfeld modules over $\mathbb{C}_\infty$ and isogenies of A-lattices of §2.3 gives rise to a diagram where arrows represent inclusion of lattices:



Proposition 2.5 further yields the following four isomorphisms of $\mathbb{F}_q$-vector spaces:

$$\Lambda_1/\Lambda \simeq \ker(\phi\to\phi/G) = G, \quad \Lambda_2/\Lambda \simeq \ker(\phi\to\phi/H) = H, \quad \Lambda_\cap/\Lambda \simeq G\cap H, \quad \Lambda_+/\Lambda \simeq G+H.$$

From the inclusion $\Lambda_\cap \subseteq \Lambda_1\cap\Lambda_2$ and the isomorphisms $\Lambda_\cap/\Lambda \simeq (\Lambda_1/\Lambda)\cap(\Lambda_2/\Lambda) \simeq (\Lambda_1\cap\Lambda_2)/\Lambda$, we deduce that $\Lambda_\cap = \Lambda_1\cap\Lambda_2$. A similar computation yields $\Lambda_+ = \Lambda_1+\Lambda_2$. The last displayed diagram simplifies to



Applying Theorem 2.6 in this situation yields

$$\log_q \mathcal{D}(\Lambda_1\cap\Lambda_2) + \log_q \mathcal{D}(\Lambda_1+\Lambda_2) = \log_q \mathcal{D}(\Lambda_1) + \log_q \mathcal{D}(\Lambda_2). \tag{5.3}$$

By construction of the A-lattices here and the definition of the local component at v of the Taguchi height (see §4.2), we have

$$\begin{aligned} h^v_{\mathrm{Tag},K}(\phi/G) &= \log_q\left[\mathcal{D}(\Lambda_1)^{-1/r}\right],\\ h^v_{\mathrm{Tag},K}(\phi/H) &= \log_q\left[\mathcal{D}(\Lambda_2)^{-1/r}\right],\\ h^v_{\mathrm{Tag},K}(\phi/(G\cap H)) &= \log_q\left[\mathcal{D}(\Lambda_\cap)^{-1/r}\right] = \log_q\left[\mathcal{D}(\Lambda_1\cap\Lambda_2)^{-1/r}\right],\\ \text{and } h^v_{\mathrm{Tag},K}(\phi/(G+H)) &= \log_q\left[\mathcal{D}(\Lambda_+)^{-1/r}\right] = \log_q\left[\mathcal{D}(\Lambda_1+\Lambda_2)^{-1/r}\right]. \end{aligned}$$

Plugging these into (5.3) concludes the proof. □

5.4. Graded height at an arbitrary place. – For any Drinfeld module ϕ defined over K and any place v of K, recall from §4.1 that

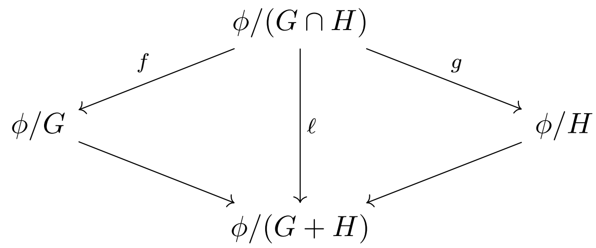
$$h^v_{\mathrm{Gr},K}(\phi) = -\min\left\{\frac{\mathrm{ord}_v(g_i)}{q^i - 1}, 1 \le i \le r\right\} = \lambda_v(\phi).$$

In this section, we prove the following result.

Theorem 5.3. *Let ϕ be a Drinfeld module defined over K, and let $G, H \in \mathtt{SG}_K(\phi)$. For any place $v \in M_K$ of K, we have*

$$h^v_{\mathrm{Gr},K}(\phi/(G \cap H)) + h^v_{\mathrm{Gr},K}(\phi/(G + H)) \le h^v_{\mathrm{Gr},K}(\phi/G) + h^v_{\mathrm{Gr},K}(\phi/H). \tag{5.4}$$

Proof. As discussed in §5.2, the data leads to a parallelogram of normalized isogenies



where f and g have trivial GCD in $K\{\tau\}$, and $\ell \in K\{\tau\}$ is the LCM of f and g.

Writing $\mu_v \coloneqq \lambda_v(\phi/(G \cap H))$ for simplicity, Proposition 4.2 yields

$$\begin{aligned} \lambda_v(\phi/G) - \lambda_v(\phi/(G \cap H)) &= V_f(\mu_v) - \mu_v, \\ \lambda_v(\phi/H) - \lambda_v(\phi/(G \cap H)) &= V_g(\mu_v) - \mu_v, \\ \text{and } \lambda_v(\phi/(G + H)) - \lambda_v(\phi/(G \cap H)) &= V_\ell(\mu_v) - \mu_v. \end{aligned}$$

Applying Theorem 3.3, we deduce that $V_\ell(\mu_v) \le V_f(\mu_v) + V_g(\mu_v) - \mu_v$. In other words, we have

$$\lambda_v(\phi/(G + H)) \le \lambda_v(\phi/G) + \lambda_v(\phi/H) - \lambda_v(\phi/(G \cap H)),$$

which is exactly the desired parallelogram inequality. □

The previous proposition also implies the following result.

Corollary 5.4. *Let ϕ be a Drinfeld module defined over K, and let $G, H \in \mathtt{SG}_K(\phi)$. For a finite place $v \in M_K^f$ of K at which ϕ has stable reduction, we have*

$$h^v_{\mathrm{Tag},K}(\phi/(G \cap H)) + h^v_{\mathrm{Tag},K}(\phi/(G + H)) \le h^v_{\mathrm{Tag},K}(\phi/G) + h^v_{\mathrm{Tag},K}(\phi/H). \tag{5.5}$$

Proof. Let $\varphi : A \to K\{\tau\}$ be a Drinfeld module defined over K. If v is a finite place of K at which φ has stable reduction, then any Drinfeld module defined over K which is isogenous to φ also has stable reduction at v (see Proposition 5.1). Moreover, we have $h^v_{\mathrm{Tag},K}(\varphi) = \lceil h^v_{\mathrm{Gr},K}(\varphi) \rceil = h^v_{\mathrm{Gr},K}(\varphi)$ by Proposition 4.1 and the definition of $h^v_{\mathrm{Tag},K}(\varphi)$. Applying this to the four Drinfeld modules in the isogeny parallelogram, the desired inequality follows from Proposition 5.3. □

5.5. Parallelogram inequalities for Drinfeld modules. – Summing up, with appropriate weights, the parallelogram inequalities for local heights obtained in the last two paragraphs, one obtains parallelogram inequalities for the global graded and Taguchi heights, as in Theorem A.

More specifically, Theorem A*(1)* immediately follows from the definition of the graded height and Theorem 5.3, while Theorem A*(2)* is a direct consequence of the definition of the Taguchi height, the equalities in Theorem 5.2, and the inequalities of Corollary 5.4.

5.6. Consequence for j-invariants in rank 2. – Let K be a finite extension of $K_0 = \mathbb{F}_q(T)$. Recall that K is equipped with an absolute logarithmic Weil height (see [GP22, §1.3]) which we denote by $\mathrm{h}_W : K \to \mathbb{Q}$. In the special case $K = K_0$, the height $\mathrm{h}_W(x)$ coincides with the degree of $x \in \mathbb{F}_q(T)$ as a rational function in T *i.e.*, $\mathrm{h}_W(x)$ is the maximum of the degrees of the numerator and denominator of x, as polynomials in T.

In this final paragraph, we restrict ourselves to considering Drinfeld modules of rank 2 defined over K. Such a Drinfeld module $\phi : A \to K\{\tau\}$ is determined by $\phi_T = T + g\,\tau + \Delta\,\tau^2 \in K\{\tau\}$, with coefficients $g \in K$ and $\Delta \in K^*$. One classically associates to ϕ its j-invariant $j(\phi) \coloneqq g^{q+1}/\Delta \in K$. One shows (see Lemma 3.8.4 in [Pap23]) that two Drinfeld modules of rank 2 defined over K are isomorphic over K^{sep} if and only if their j-invariants are equal. Moreover, for any $j \in K$, there exists a Drinfeld module $\phi : A \to K\{\tau\}$ such that $j(\phi) = j$. One then defines the *modular height* of ϕ by:

$$h_{\mathrm{m}}(\phi) \coloneqq \mathrm{h}_W(j(\phi)) = \frac{1}{[K : K_0]} \sum_{v \in M_K} f_v \max\{-\mathrm{ord}_v(j(\phi)), 0\}.$$

A quick computation using the product formula for K proves (see also [BPR21, §2.2]) that

$$h_{\mathrm{m}}(\phi) = (q^2 - 1)\, h_{\mathrm{Gr},K}(\phi).$$

Our Theorem A*(1)* can thus be rewritten as follows: for any $G, H \in \mathtt{SG}_K(\phi)$, we have

$$h_{\mathrm{m}}(\phi/(G \cap H)) + h_{\mathrm{m}}(\phi/(G + H)) \leq h_{\mathrm{m}}(\phi/G) + h_{\mathrm{m}}(\phi/H). \tag{5.6}$$

Together with the preceding remarks, this further specializes to Corollary C in the introduction when $K = K_0$. We illustrate this result by a worked out example.

Example 5.5. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of rank 2 defined over K. Consider its T-torsion "subgroup" $\phi[T] \coloneqq \ker(\phi_T)$ (see [Pap23, §3.5]). Clearly, one has $\phi[T] \in \mathtt{SG}_K(\phi)$ and $\dim_{\mathbb{F}_q} \phi[T] = \deg_\tau \phi_T = 2$. The polynomial $f = T^{-1} \cdot \phi_T = 1 + g\,T^{-1}\tau + \Delta\,T^{-1}\tau^2 \in K\{\tau\}$ is the unique normalized polynomial with kernel $\phi[T]$. A straighforward computation shows that f is the "quotient isogeny" $\phi \to \psi \coloneqq \phi/\phi[T]$, where ψ is given by $\psi_T = T + (g\,T^{q-1})\,\tau + (\Delta\,T^{q^2-1})\,\tau^2$. Note that $T \cdot \psi_T \cdot T^{-1} = \phi_T$ so that $\psi \simeq \phi$.

We now pick a basis $(e_1, e_2) \in K^2$ of $\phi[T]$ as a $\mathbb{F}_q$-vector space, and let $G_1 \coloneqq \mathbb{F}_q\, e_1$, $G_2 \coloneqq \mathbb{F}_q\, e_2$. By construction, G_1, G_2 are elements of $\mathtt{SG}_K(\phi)$, they have trivial intersection, and their (direct) sum is $\phi[T]$. Applying the parallelogram inequality (5.6), we get

$$2\,h_{\mathrm{m}}(\phi) = h_{\mathrm{m}}(\phi) + h_{\mathrm{m}}(\phi/\phi[T]) = h_{\mathrm{m}}(\phi/(G_1 \cap G_2)) + h_{\mathrm{m}}(\phi/(G_1 + G_2)) \leq h_{\mathrm{m}}(\phi/G_1) + h_{\mathrm{m}}(\phi/G_2).$$

In other words, we have

$$\frac{h_{\mathrm{m}}(\phi/G_1) + h_{\mathrm{m}}(\phi/G_2)}{2} \geq h_{\mathrm{m}}(\phi).$$

Acknowledgments – The authors are grateful to Francesco Campagna for helpful comments on this paper. Most of this project was completed during a *GandA Research In Pairs* visit of RG to LB. This visit was made possible through funding from IRN GandA (CNRS), which the authors heartily thank. RG wishes to thank Stellenbosch University for its hospitality during his stay. RG is partially supported by ANR-23-CE40-0006-01 *GAEC*, and FP by ANR-20-CE40-0003 *Jinvariant*.

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